Brill-Noether loci of rank 2 vector bundles over an algebraic curve

Rayfield, Alan Christopher

Brill-Noether loci of rank 2 vector bundles over an algebraic curve

by

Alan Christopher Rayfield

A thesis presented for
the degree of Doctor of Philosophy
August 1999

Department of Mathematical Sciences,
University of Durham,
South Road, Durham.
DH1 3LE England

Abstract

Brill-Noether loci of rank 2 vector bundles over an algebraic curve

Alan Christopher Rayfield

In this thesis the Brill-Noether loci $\mathcal{W}^r$ of rank 2 stable vector bundles of canonical determinant over an algebraic curve are studied. We analyse three conjectures on the nonexistence, dimension and smoothness of $\mathcal{W}^r$, collectively known as the Brill-Noether conditions. The local structure of $\mathcal{W}^r$ is described by a symmetric Petri map; assuming that $\mathcal{W}^r \neq \emptyset$, the injectivity of this map ensures the dimension and smoothness conditions we are aiming for. The nonemptiness of $\mathcal{W}^r$ is shown by constructing the appropriate bundles from extensions of line bundles. In a similar vein the nonexistence conjecture is addressed by showing that certain bundles are extensions of line bundles that are prohibited on the curve. Finally, subject to an assumption, the Petri map is shown to be injective for genus ≤ 10; which allows us to prove that the Brill-Noether conditions hold for genus ≤ 10, improving on the genus ≤ 7 results of Bertram-Feinberg [4].

Preface

Brill-Noether loci of rank 2 vector bundles over an algebraic curve

Alan Christopher Rayfield

This thesis is the result of research carried out between October 1995 and August 1999 under the supervision of Dr W.M.Oxbury. The work presented in thesis has not been submitted for any other degree either at Durham or any other University.

Throughout this thesis all non-original material, either in the form of statement of proof, is referred to its original source.

Acknowledgements

I would like to thank my supervisor Bill Oxbury for all the insightful mathematical advice he has given throughout my studies at Durham.

Thanks go to my family and all those whose support and friendship made my time in Durham frivolous, enjoyable and above all rewarding. However, of these people a special mention must be given to Sarah and Akay, whose influence changed my life.

This thesis is dedicated to Dyfrig Williams.

Contents

Chapter 1

Introduction

This thesis studies the Brill-Noether loci of rank 2 semistable bundles with canonical determinant. These are the subvarieties:

$$\mathcal{W}^r = \{E \in \mathcal{SU}(2,K) \mid E \text{ is stable and } h^0(C,E) \geq r+1\}.$$

The approach here is to look at the $\mathcal{W}^r$ from the perspective of three conjectures made by Bertram-Feinberg [4]. They are given in terms of the Brill-Noether number which is defined to be:

$$\rho(r) := 3g - 3 - \binom{r+2}{2}.$$

C is a generic curve of genus g, then the following properties of $\mathcal{W}^r$ hold:

- *Nonexistence : If $\rho(r) < 0$ then $\mathcal{W}^r = \emptyset$;*
- *Dimension : If $\rho(r) \geq 0$ then dim $\mathcal{W}^r = \rho(r)$;*
- *Smoothness : $\mathcal{W}^r$ is smooth away from $\mathcal{W}^{r+1}$.*

These properties have been shown to hold for $g \leq 7$. Building on this work, the main result of the thesis is to prove them for $g \leq 10$.

In the following sections these conjectures are motivated and are seen to arise through the analogy with the Brill-Noether theory of line bundles. It is shown that the central features of verifying the conjectures are, first to ascertain that $\mathcal{W}^r \neq \emptyset$ (since no existence theorem exists at present) and secondly that the Petri map:

$$\mu : \mathrm{Sym}^2 H^0(C, E) \to H^0(C, \mathrm{Sym}^2 E),$$

is injective for all $E \in \mathcal{W}^r - \mathcal{W}^{r+1}$.

Chapter 2 reviews some necessary theory (section 2.1) and then addresses the existence of $\mathcal{W}^r$ and subsequently the nonexistence conjecture.

Chapter 3 deals with the question of injectivity of the Petri map and generalises the work of Bertram-Feinberg [4]. Chapter 4 contains the study of a technical condition required in Chapter 3.

Finally Chapter 5 looks at a specific example, the Brill-Noether locus $\mathcal{W}^5$ for a genus 9 curve.

1.1 Brill-Noether theory of line bundles

The Brill-Noether problem is the question of what special linear series exist on a generic algebraic curve. The body of theory answering this question is known as the Brill-Noether theory. The motivation for studying the Brill-Noether theory of line bundles is the correspondence between linear series and maps to projective space:

$$\{g^r_d s\} \leftrightarrow \{f : C \to \mathbb{P}^r\},$$

where we use the standard notation that a g^r_d is a linear subseries $\mathcal{D} \subseteq |D|$ for some divisor D, where $\deg(D) = d$ and $\dim(\mathcal{D}) = r$. We recall here that a linear series $\mathcal{D}$ is special if it satisfies the conditions $\dim(\mathcal{D}) = r \geq 0$ and $h^1(C, D) > 0$.

The first natural object to look at is:

$$G^r_d = \{ \text{ linear series } \mathcal{D} \mid \mathcal{D} \text{ is a } g^r_d\}.$$

Complete linear series are in one to one correspondence with line bundles so we also look at the Brill-Noether locus:

$$W^r_d = \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geq r + 1\}.$$

The Brill-Noether locus is a determinantal subscheme of $\mathrm{Pic}^d(C)$ (for the construction see Arbarello-Cornalba-Griffiths-Harris [1] page 176). The space of linear series G^r_d can be shown to be the canonical blow-up of W^r_d along the subvariety W^{r+1}_d. Aside from their importance to projective realisations of curves, G^r_d and W^r_d also enjoy interesting geometry in themselves (for example see [1] chapter 5).

In the study of the Brill-Noether theory the tool used is the Petri map, whose importance derives from its description of the Zariski tangent space of W^r_d at a point. Given $L \in W^r_d - W^{r+1}_d$ the Petri map is the cup product homomorphism:

$$\mu : H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K). \tag{1.1}$$

Denoting the degree d Jacobian of the curve C by J^d_C, we recall that the cotangent space of J^d_C at a point L is the space of sections of the canonical bundle: $T^*_L J^d_C = H^0(C, K)$. In particular $\dim(H^0(C, K)) = h^0(C, K) = g$, the genus of the curve. Furthermore it can also be shown (see Arbarello-Cornalba-Griffiths-Harris [1] Proposition 4.2 page 189) that $\mathrm{im}(\mu)^\perp = T_L W^r_d$. Here the perpendicular space is defined by the Serre duality pairing $H^1(C, L) \otimes H^0(C, KL^{-1}) \to \mathbb{C}$. These two results together give:

$$\begin{aligned}\dim(T_L W^r_d) &= \mathrm{im}(\mu)^\perp = \dim\,(T_L J) - \mathrm{rank}(\mu) = h^0(C, K) - \mathrm{rank}(\mu)\\ &= g - \mathrm{rank}(\mu).\end{aligned} \tag{1.2}$$

Noting by the Riemann-Roch formula that $h^1(C, L) = g - d - 1 + h^0(C, L) = g - d + r$, we have the following upper bound on the rank of μ:

$$\begin{aligned}\mathrm{rank}(\mu) &\leq \dim(H^0(C, L) \otimes H^0(C, KL^{-1})) = h^0(C, L)h^0(C, KL^{-1})\\ &\leq (r+1)(g - d + r),\end{aligned} \tag{1.3}$$

with equality when μ is injective. Equations (1.2) and (1.3) give us a lower bound on the dimension of the Zariski tangent space:

$$\dim(T_L W^r_d) \geq g - (r+1)(g - d + r). \tag{1.4}$$

To get a good estimate of $\dim(W_d^r)$ we would hope that $\dim(W_d^r) = \dim(T_L W_d^r)$, which occurs if and only W_d^r is smooth at L. Furthermore to have equality in equation (1.4) we require μ to be injective. Assuming both smoothness and injectivity conditions we obtain:

$$\dim(W_d^r) = g - (r+1)(d-g+r). \tag{1.5}$$

This is the "expected" dimension of W_d^r calculated by determinantal means, see [1] page 181. We now make the following definition.

Definition 1.1.1. *For a curve C of genus g and integers $r \geq 0$ and $d > 0$ the Brill-Noether number is:*

$$\rho(r,d) = g - (r+1)(g-d+r).$$

The Brill-Noether number is significant in the theory as can be seen from the following classical results in Brill-Noether theory.

Theorem 1.1.2. *([1] page 206) Let C be a smooth curve of genus g. Let d and r be integers such that $d \geq 1, r \geq 0$; if $\rho(r,d) \geq 0$ then G_d^r and hence W_d^r are nonempty. Furthermore, every component of G_d^r has dimension at least $\rho(r,d)$; the same is true for W_d^r provided $r \geq d-g$.*

Theorem 1.1.3. *([1] page 212) Let C be a smooth curve of genus g. Let r and d be integers such that $d \geq 1, r \geq 0$. Assuming that $\rho(r,d) > 0$, then G_d^r and hence W_d^r are connected.*

If C is a generic curve though, we have a strong result about the Petri map which tells us that W_d^r satisfies all the properties that we would like.

Theorem 1.1.4. *([1] page 215)Let C be a generic curve of genus g, then the Petri map μ is injective for all $L \in W_d^r$.*

Corollary 1.1.5. *If C is a generic curve and $\rho(r,d) < 0$ then $W_d^r = G_d^r = \emptyset$.*

Proof. Suppose for a contradiction that $W_d^r \neq \emptyset$, pick a bundle $L \in W_d^r$, then $r(L) = R \geq r$. By Theorem 1.1.4 μ is injective at L, so equation (1.4) becomes $\dim(T_L W_d^R) = \rho(R,d) \leq \rho(r,d) < 0$. Therefore the tangent space is empty at L, contradicting the existence of L.

To show emptiness of G_d^r, note that a linear system $\mathcal{D} \in G_d^r$ gives rise to a bundle $L \in W_d^R$ for some $R \geq r$. However, $\rho(R,d) \leq \rho(r,d) < 0$ so by the above $W_d^R = \emptyset$. $\square$

Corollary 1.1.6. *If C is a generic curve with $\rho(r,d) \geq 0$ then $\dim(W_d^r) = \rho(r,d)$ and W_d^r is smooth away from W_d^{r+1}.*

Proof. By Theorem 1.1.2, W_d^r is non-empty with $\dim(W_d^r) \geq \rho(r,d)$. Pick $L \in W_d^r - W_d^{r+1}$, then Theorem 1.1.4 tells us that the map μ is injective at L. By equation (1.4) the dimension of the tangent space is equal to the Brill-Noether number, which gives:

$$\rho(r,d) = \dim(T_L W_d^r) \geq \dim(W_d^r) \geq \rho(r,d).$$

This gives the results $\dim(W_d^r) = \rho(r,d)$ and W_d^r is smooth away from W_d^{r+1} because $\dim(T_L W_d^r) = \dim(W_d^r)$. $\square$

Corollary 1.1.7. *If C is a generic curve and $\rho(r,d) \geq 0$ then G_d^r is smooth of pure dimension $\rho(r,d)$.*

Proof. Let $w \in G_d^r$ correspond to an $r+1$ dimensional subspace W of $H^0(C,L)$ where L is the line bundle associated with W. We may define the following map by the cup product:

$$\mu_W : W \otimes H^0(C, KL^{-1}) \to H^0(C,K).$$

In an analysis similar to that for μ (see [1] page 187) we have $\dim(T_W G_d^r) = \rho(r,d) + \dim(\ker(\mu_W))$. Clearly if μ is injective then so too is μ_W. Again by Theorem 1.1.2 $\dim(T_W G_d^r) = \dim(G_d^r) = \rho(r,d)$, so G_d^r is smooth and of the correct dimension. $\square$

Corollary 1.1.8. *([1] page 214) If C is a generic curve and $\rho(r,d) \geq 1$ then G_d^r and W_d^r are irreducible.*

The following theorem of Castelnuovo gives a useful result for generic curves.

Theorem 1.1.9. *Suppose that C is a generic curve of genus g. If $\rho(r,d) = 0$ then the number of g^r_d's is:*

$$g!\prod_{i=0}^{r}\frac{i!}{(g-d+r+i)!}.$$

Before finishing this section on line bundles we remark that the theta divisor:

$$\Theta := \{L \in J^{g-1}_C \mid h^0(C,L) \neq 0\},$$

is identical to the Brill-Noether locus W^0_{g-1}.

1.2 Background on rank 2 vector bundles

In this section we review some background on rank 2 vector bundles. To start we recall the definitions of stability and S-equivalence that are required in the construction of the moduli space $\mathcal{U}(n,d)$ of rank n, degree d semistable bundles. The subspace $\mathcal{SU}(n,L)$ of semistable rank n bundles with fixed determinant L is introduced. We then focus on $\mathcal{SU}(2,K)$, the space we work in for the remainder of the thesis. Some results on the geometry of $\mathcal{SU}(2,K)$ are then stated that will be used later.

To construct a moduli space of rank n bundles we need to restrict attention to semistable vector bundles. This notion is defined in terms of the slope of a bundle, which is given by $\mu(E) := \deg(E)/\mathrm{rank}(E)$. Then E is stable (resp. semistable) if for all proper subbundles $F \subset E$;

$$\mu(F) < \mu(E)(\text{resp. } \leq).$$

In this thesis bundles will be referred to as unstable if they are not semistable. More precisely E is unstable if there exists a proper subbundle F such that $\mu(F) > \mu(E)$.

With the above definition a moduli space of stable bundles may be constructed, which is a quasi-projective variety. The moduli space may be compactified by adding in semistable bundles under the following equivalence relation. For E a semistable bundle we have a sequence of subbundles of the form

$$0 = E_m \subset E_{m-1} \subset \cdots \subset E_1 \subset E_0 = E,$$

such that $E_m, \frac{E_{m-1}}{E_m}, \ldots, \frac{E_0}{E_1}$ are stable and:

$$\mu(E_m) = \mu(E_{m-1}/E_m) = \cdots = \mu(E_0/E_1) = \mu(E_0). \tag{1.6}$$

For each E the set of bundles $\{E_m, \frac{E_{m-1}}{E_m}, \ldots, \frac{E_0}{E_1}\}$ is unique (up to isomorphism) although the filtration (1.6) may not be. This result is the analogue of the Jördan-Holder theorem for vector bundles. We may now define the graded bundle :

$$\mathrm{gr}(E) := E_m \oplus E_{m-1}/E_m \oplus \cdots \oplus E_0/E_1,$$

and finally our S-equivalence relation. If E and E' are two rank n degree d bundles, then they are S-equivalent if and only if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$.

For a stable vector bundle E the filtration (1.6) above becomes $0 \subset E$ since there are no subbundles F such that $\mu(F) = \mu(E)$. For a stable bundle E, we have $E = \mathrm{gr}(E)$; so two stable bundles E and E' are S-equivalent if and only if $E \cong E'$. Hence the moduli space constructed in the following theorem contains all the stable bundles as single point equivalence classes.

Theorem 1.2.1. *There is a normal, irreducible projective variety $\mathcal{U}(n,d)$ which parametrises the set of S-equivalence classes of semistable bundles of rank n and degree d. The dimension of $\mathcal{U}(n,d)$ is $n^2(g-1)+1$, where $g \geq 2$.*

This result was first shown by Mumford; for a proof we refer to Newstead [22] Theorem 5.8 page 143. From now on we will restrict attention to curves of genus $g \geq 3$.

By taking the determinants of bundles in the moduli space we obtain the natural holomorphic map to the Jacobian:

$$\det : \mathcal{U}(n,d) \to J_C^d.$$

We must be careful that the determinant map is well defined over the equivalence class; this can be seen by noting that for $E, F \in \mathcal{U}(n,d)$ we have $\mathrm{gr}(E) = \mathrm{gr}(F)$ which implies that $\det(E) = \det(F)$. We now define

$$\mathcal{SU}(n,L) := \det^{-1}(L) \subset \mathcal{U}(n,d),$$

which is the coarse moduli space of rank n semistable bundles with fixed determinant L. A coarse moduli space in this case means that for any family $\mathcal{V} \to C \times S$ of vector bundles of rank n and degree d on C, where the base space S is an algebraic variety then the map

$$S \xrightarrow{\epsilon} \mathcal{U}(n,d)$$
$$s \mapsto \mathcal{V}|_{C\times\{s\}},$$

is a morphism of algebraic varieties. The moduli space $\mathcal{U}(n,d)$ is, up to a finite étale covering, the product of $\mathcal{SU}(n,L)$ with the Jacobian. Therefore $\dim(\mathcal{SU}(n,L)) = n^2(g-1)+1-g$.

With the above preamble on rank n vector bundles in place we may consider our particular case of rank 2 vector bundles. It is worth noting that for all $M \in \mathrm{Pic}(C)$ there is an isomorphism $\mathcal{SU}(2,L) \xrightarrow{\sim} \mathcal{SU}(2,LM^2)$ given by $E \mapsto E \otimes M$. Consequently there are precisely 2 isomorphism classes: $\mathcal{SU}(2,\mathrm{odd})$ and $\mathcal{SU}(2,\mathrm{even})$.

In looking at the even case the most natural determinant to consider is either $\mathcal{O}$ or K. In the remainder of the thesis study is restricted to $\mathcal{SU}(2,K)$. Choosing this determinant means that $\chi(E) = h^0(C,E) - h^1(C,E) = \deg(E) - 2(g-1) = 0$; which allows us to canonically define a theta divisor on $\mathcal{SU}(2,K)$ without requiring a twist by a line bundle (see below).

We now make a definition of the theta divisor on $\mathcal{SU}(2,K)$, analogous to that on J_C^{g-1}:

$$\Theta_2 = \overline{\{E \in \mathcal{SU}(2,K) \mid E \text{ stable, and } h^0(C,E) \neq 0\}}. \tag{1.7}$$

In fact this turns out to be a Cartier divisor, see Drezet-Narasimhan [7] (the important point being that $\chi(E) = 0$; in $\mathcal{SU}(2,\mathcal{O})$ the condition becomes $h^0(C, E\otimes L) \neq 0$

for some $L \in J_C^{g-1}$).

The determinantal line bundle on $\mathcal{SU}(2,K)$ is defined to be $\mathcal{L} = \mathcal{O}(\Theta_2)$. In fact Drezet-Narasimhan have shown [7] (Theoreme B)that the Picard group of $\mathcal{SU}(2,K)$ is generated by this line bundle:

$$\mathrm{Pic}(\mathcal{SU}(2,K)) = \mathbb{Z}\mathcal{L}.$$

The bundle $\mathcal{L}$ gives a map $\phi_{\mathcal{L}} : \mathcal{SU}(2,K) \to |\mathcal{L}|^{\star}$. It has been shown by Beauville [2] (Theoreme 1)that $|\mathcal{L}| \cong |2\Theta| \cong \mathbb{P}^{2^g-1}$. This result has subsequently been generalised to rank n vector bundles by Beauville-Narasimhan-Ramanan [3]; the statement becoming $|\mathcal{L}| \cong |n\Theta|$. It has recently been shown by van Geeman-Izadi [9] (Theorem 3) that $\phi_{\mathcal{L}}$ embeds $\mathcal{SU}(2,K)$ into $|2\Theta|^{\star}$.

We now look at the part the Jacobian has to play in the study of $\mathcal{SU}(2,K)$. The Kummer variety is defined to be the quotient $\mathcal{K}_C = J^{g-1}/\sim$ where the relation $\sim$ is given by:

$$L \sim L' \Leftrightarrow L' = L \text{ or } L' = KL^{-1}.$$

We may define a map $\kappa : \mathcal{K}_C \to \mathcal{SU}(2,K)$ where $\kappa : L \mapsto L \oplus KL^{-1}$. Clearly κ embeds $\mathcal{K}_C$ in $\mathcal{SU}(2,K)$.

The image of $\phi_{\mathcal{L}}$ restricted to $\kappa(\mathcal{K}_C)$ is the image of $\mathcal{K}_C$ in $|2\Theta|^{\star}$ under the linear system $|2\Theta|$. Thus we have the commutative diagram:

$$\begin{tikzcd}
\mathcal{K}_C \arrow[r, "\kappa"] \arrow[dr, "|2\Theta|"'] & \mathcal{SU}(2,K) \arrow[d, "\phi_{\mathcal{L}}"] \\
& |2\Theta|^{\star}
\end{tikzcd}$$

The image of the Kummer variety $\mathcal{K}_C$ under the map κ is the semistable boundary of $\mathcal{SU}(2,K)$, that is to say the strictly semistable bundles on C. Moreover, the stable bundles form a smooth open subset of $\mathcal{SU}(2,K)$.

1.3 Brill-Noether loci of rank 2 vector bundles

For line bundles there are a number of theorems in section 1.1 which give us a good understanding of the varieties W^r_d and G^r_d and hence the linear series on the curve. We now define the Brill-Noether locus of rank 2 vector bundles on a curve by analogy with the line bundle subvariety W^r_d.

Definition 1.3.1. *The Brill-Noether locus of rank 2 vector bundles is defined to be:*

$$\mathcal{W}^r := \{E \in \mathcal{SU}(2,K) \mid E \text{ is stable}, h^0(C,E) \geq r+1\}.$$

$\mathcal{W}^r$ may be constructed as a determinantal variety in a similar manner to the line bundle case, see Laszlo [14] section 2. Recalling our definition of the theta divisor (1.7) on $\mathcal{SU}(2,K)$ we see that $\Theta_2 = \overline{\mathcal{W}^0}$ - the closure of $\mathcal{W}^0$. Note the similarity with the line bundle case where $\Theta = W^0_{g-1}$.

We now try to imitate the procedure for line bundles by calculating the expected dimension of $\mathcal{W}^r$ using a Petri map (c.f. (1.1)). Let $E \in \mathcal{W}^r - \mathcal{W}^{r+1}$ then the Petri map is defined by the cup product homomorphism:

$$\mu_0 : H^0(C,E) \otimes H^0(C,E^\star \otimes K) \to H^0(C, E \otimes E^\star \otimes K).$$

Since $\det(E) = K$ we have that $E^\star \otimes K \cong E$. The map then becomes:

$$\mu_0 : H^0(C,E) \otimes H^0(C,E) \to H^0(C, E \otimes E^\star \otimes K).$$

The domain of μ_0 is the tensor square and the map itself is given by multiplication of sections so we can see that $\mu_0(\bigwedge^2 H^0(C,E)) = 0$. We therefore factor out $\bigwedge^2 H^0(C,E)$ to obtain a Petri map:

$$\mu : \mathrm{Sym}^2 H^0(C,E) \to H^0(C, E \otimes E^\star \otimes K).$$

We now give a geometric interpretation of the image. The Zariski tangent space to $\mathcal{U}(2, 2g-2)$ at E is:

$$T_E\mathcal{U}(2, 2g-2) = H^1(C, \mathrm{End}(E)) \cong H^0(C, E \otimes E^\star \otimes K)^\star.$$

However, $\mathrm{End}(E) = \mathrm{End}_0(E) \oplus \mathcal{O}$, where $\mathrm{End}_0(E)$ denotes the tracefree endomorphisms. Therefore $H^0(C, E \otimes E^\star \otimes K)^\star = H^1(C, \mathrm{End}_0(E)) \oplus H^1(C, \mathcal{O}) = T_E\mathcal{SU}(2, K) \oplus T_0 J_C^0$. The bundle E has fixed determinant K so the image of μ is contained in the cotangent space of $\mathcal{SU}(2, K)$ which is $H^1(C, \mathrm{End}_0(E))^\star \cong H^0(C, K \otimes \mathrm{End}_0))$. However, $H^0(C, K \otimes \mathrm{End}_0(E)) \cong H^0(C, \mathrm{Sym}^2 E)$, so the Petri map can then be rewritten in a symmetric form as:

$$\mu : \mathrm{Sym}^2 H^0(C, E) \to H^0(C, \mathrm{Sym}^2 E). \tag{1.8}$$

By the construction of $\mathcal{W}^r$ as a variety (see Laszlo [14] Lemme II.5), $T_E\mathcal{W}^r = (\mathrm{im}\mu)^\perp$. Furthermore, the dimension of $\mathrm{Sym}^2 H^0(C, E)$ is $\binom{r+2}{2}$, since $h^0(C, E) = r+1$. Suppose that $\mathcal{W}^r$ is smooth at E so the dimension of the Zariski tangent space and the dimension of $\mathcal{W}^r$ coincide. Moreover, assume μ is injective, forcing the image of μ to have dimension $\binom{r+2}{2}$; then combining these facts gives:

$$3g-3-\binom{r+2}{2} = \dim(\mathcal{SU}(2, K)) - \mathrm{rank}(\mu) = \dim(\mathrm{im}\mu)^\perp = \dim(T_E\mathcal{W}^r) = \dim(\mathcal{W}^r).$$

The number just calculated is the "expected dimension" of $\mathcal{W}^r$. This motivates the following definition.

Definition 1.3.2. *The Brill-Noether number for vector bundles in $\mathcal{SU}(2, K)$ is:*

$$\rho(r) = 3g - 3 - \binom{r+2}{2}.$$

The Brill-Noether number $\rho(r)$ will be a useful tool in studying Brill-Noether theory of rank 2 bundles just as the Brill-Noether number $\rho(r, d)$ for line bundles turned out to be. Hitherto the description of the Brill-Noether loci for rank 2 vector bundles by the Petri map has been analogous to that of the line bundle case. For this reason Bertram-Feinberg [4] made the following conjectures suggested by the line bundle case, see Corollary 1.1.5 and Corollary 1.1.6.

Conjecture 1.3.3. *Let C be a generic curve of genus g, then the following properties of $\mathcal{W}^r$ hold:*

- *Nonexistence* : *If $\rho(r) < 0$ then $\mathcal{W}^r = \emptyset$;* (1.9)
- *Dimension* : *If $\rho(r) \geq 0$ then dim $\mathcal{W}^r = \rho(r)$;* (1.10)
- *Smoothness* : *$\mathcal{W}^r$ is smooth away from $\mathcal{W}^{r+1}$.* (1.11)

These will be refered to as the Brill-Noether conditions.

These conjectures seem credible in the light of the following result (for a proof see Mukai [18]).

Theorem 1.3.4. *Let C be an algebraic curve. If $\mathcal{W}^r \neq \emptyset$ and $\rho(r) \geq 0$ then dim$(\mathcal{W}^r) \geq \rho(r)$.*

For comparison, consider the result for line bundles (Theorem 1.1.2); although both hold for all curves an important distinction is the lack of an existence result in the rank 2 case.

Some work has been done in verifying the Brill-Noether conditions, Bertram-Feinberg [4] showed the conjectures to hold for $g \leq 7$.

It is the aim of this thesis to show that the Brill-Noether conditions hold for $g \leq 10$.

To prove (1.10) and (1.11), a general method can be constructed. Firstly the Brill-Noether locus $\mathcal{W}^r$ is shown to be nonempty for $\rho(r,d) \geq 0$, which enables us to use the lower bound established in Theorem 1.3.4. The next step is to show that μ is injective for $E \in \mathcal{W}^r - \mathcal{W}^{r+1}$. This will be enough to show our two conditions, as we now explain. By Theorem 1.3.4, if $\mathcal{W}^r \neq \emptyset$ then $\dim(\mathcal{W}^r) \geq \rho(r)$. The Petri map is injective for all $E \in \mathcal{W}^r - \mathcal{W}^{r+1}$ so $\mu : \mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E)$ has rank equal to $\dim(\mathrm{Sym}^2 H^0(C,E)) = \binom{r+2}{2}$. Note that $\dim(\mathcal{SU}(2,K)) = 3g-3$ and recall that $(\mathrm{im}(\mu))^\perp = T_E\mathcal{W}^r$. Combining these facts with the rank of μ and the lower bound on $\mathcal{W}^r$ gives:

$$\rho(r) = 3g - 3 - \binom{r+2}{2} = \dim(\mathrm{im}(\mu)^\perp) = \dim(T_E\mathcal{W}^r) \geq \dim(\mathcal{W}^r) \geq \rho(r).$$

Hence $\dim(\mathcal{W}^r) = \rho(r)$ and $\mathcal{W}^r$ is smooth at E.

In chapter 2 the nonemptiness of $\mathcal{W}^r$ for $\rho(r) \geq 0$ is considered, the result is the following.

Proposition 1.3.5. *Let C be a generic curve of genus $g \leq 10$. If $\rho(r) \geq 0$ then $\mathcal{W}^r \neq \emptyset$.*

In the final section of this chapter the condition (1.9) is addressed, the nonexistence statement is given by

Proposition 1.3.6. *Let C be a generic curve of genus $g \leq 11$ or $g = 13$. If $\rho(r) < 0$, then $\mathcal{W}^r = \emptyset$.*

Most of the thesis is concerned with injectivity of the Petri map and this occupies the whole of chapters 3 and 4. The conclusion of these sections is our main result.

Theorem 1.3.7. *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2,K)$ stable with $h^0(C,E) \leq 6$. Then the Petri map:*

$$\mu : Sym^2 H^0(C,E) \to H^0(C, Sym^2 E),$$

is injective.

Unfortunately we can only prove this result subject to an assumption (Assumption 4.2.1).

However the nonexistence result 1.3.6 tells us that for genus $g \leq 10$ the number of sections never exceeds 6, since $\rho(6) < 0$ for $g \leq 10$. Therefore Theorem 1.3.7 implies that conditions 1.10 and 1.11 hold for $g \leq 10$. Together with Proposition 1.3.6 we have proved that the Brill-Noether conditions hold for $g \leq 10$.

In the last chapter we study the geometry of the Brill-Noether locus $\mathcal{W}^5$ for generic genus 9 curves.

Chapter 2

Existence properties of Brill-Noether loci

This chapter has as its goal existence properties of the Brill-Noether loci. In the next section we follow Oxbury-Pauly-Previato [23] in setting up our approach to studying rank 2 vector bundles in terms of extensions of line bundles. A series of results is developed that will be required throughout the thesis. The following section looks at the condition $\mathcal{W}^r \neq \emptyset$, for $\rho(r) \geq 0$ where $3 \leq g \leq 10$. For genus 8 the structure of the Brill-Noether locus is described more closely. In the final section the nonexistence condition (1.9) is considered.

2.1 Results on extension spaces

Line bundles may be studied in terms of divisors on the curve; however rank 2 bundles have no such convenient characterisation. In order to construct rank 2 bundles lying in the Brill-Noether loci we study the spaces $\mathbb{P}\mathrm{Ext}^1(K - D, D)$ of extensions

$$0 \to \mathcal{O}(D) \to E \to K(-D) \to 0, \tag{2.1}$$

for some divisor D of degree d. We will take D to be effective since E will have sections if and only if it is such an extension with D effective. In this thesis we are concerned with bundles lying in $\overline{\mathcal{W}^r} \subset \mathcal{SU}(2,K)$; in particular they are semistable. For this reason we only consider $\mathbb{P}\mathrm{Ext}^1(K-D,D)$ where $d \leq g-1$, as otherwise sequence (2.1) would give an unstable extension. If we have equality, then $\mathrm{gr}(E) = \mathcal{O}(D) \oplus K(-D)$, so E is S-equivalent to $\mathcal{O}(D) \oplus K(-D)$. Conversely $\mathrm{gr}(E) = \mathcal{O}(D) \oplus K(-D)$ implies that $\deg(D) = g-1$. Hence $d = g-1$ if and only if E is S-equivalent to $\mathcal{O}(D) \oplus K(-D)$.

We now define the Clifford index (see Green-Lazarfeld [10]) associated to divisors on the curve C. The Clifford index of a divisor D is Cliff(D) =deg(D) $-$ $2r(D)$ and the Clifford index of the curve is $\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(D) \mid r(D), r(K(-D)) \geq 1\}$. Green and Lazarsfeld [10] give an upper bound on the Clifford index:

$$\mathrm{Cliff}(C) \leq \left[\frac{g-1}{2}\right] \text{ with equality if C is generic.} \tag{2.2}$$

The Clifford index will give us a good way of determining whether certain divisors exist on a given generic curve.

The space $\mathbb{P}\mathrm{Ext}^1(K-D,D)$ parametrises isomorphism classes of nonsplit extensions of (2.1), so there is rational coarse moduli map ϵ_D from the extension space to the moduli space. By using Serre duality we can say that $\mathrm{Ext}^1(K-D,D) \cong H^1(2D-K) = H^0(2K-2D)^\star$. We then get the sequence of maps:

$$C \xrightarrow{|2K-2D|} \mathbb{P}\mathrm{Ext}^1(K-D,D) \cong \mathbb{P}^{3g-4-2d} \xrightarrow{\epsilon_D} \mathcal{SU}(2,K). \tag{2.3}$$

The structure of $\mathbb{P}\mathrm{Ext}^1(K-D,D)$ is now described; in particular those extensions that give rise to bundles in $\mathcal{W}^r$ are identified. If we consider the element $e \in \mathbb{P}\mathrm{Ext}^1(K-D,D)$ corresponding to:

$$0 \to \mathcal{O}(D) \to E \to K(-D) \to 0,$$

then by identifying $H^0(C,K(-D))$ and $H^1(C,D)^\star$ using Serre duality the coboundary map of the long exact sequence in cohomology is:

$$\delta(e) : H^1(C,D)^\star \to H^1(C,D).$$

Moreover $\delta(e)^* : H^1(C,D)^* \to H^1(C,D)$ is the same map, $\delta(e)$. Therefore $\delta(e)$ is symmetric, that is an element of $\mathrm{Sym}^2 H^1(C,D)$. Thus we have a map $\delta : \mathrm{Ext}^1(K-D,D) \to \mathrm{Sym}^2 H^1(C,D)$; by abuse of language, we denote also by δ the induced map $\delta : \mathbb{P}\mathrm{Ext}^1(K-D,D) - \mathbb{P}\ker(\delta) \to \mathrm{Sym}^2 H^1(C,D)$.

The space $\mathbb{P}\mathrm{Sym}^2 H^1(C,D)$ may be stratified by the rank of the symmetric maps. On the other hand the space may be stratified by the secants of $\mathbb{P}H^1(C,D)$ embedded in $\mathbb{P}\mathrm{Sym}^2 H^1(C,D)$ by the Veronese map:

$$\mathrm{Ver} : \mathbb{P}H^1(C,D) \to \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \text{ where } \mathrm{Ver} : \eta \mapsto \eta \otimes \eta.$$

We would hope that the two stratifications coincide, in fact they may be identified in the following way:

$$\mathrm{Sec}^n(\mathrm{Ver}\mathbb{P}H^1(C,D) = \{a \in \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \mid \mathrm{rank}(a) \leq n\}$$

where $n = 1, \ldots, h^1(C,D)$. The map δ is the dual of the multiplication map $\mathrm{Sym}^2 H^0(C, K-D) \to H^0(C, 2K-2D)$, so we may construct the following commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{|K-D|} & \mathbb{P}H^1(C,D) \\
{\scriptstyle |2K-2D|}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}} \\
\mathbb{P}\mathrm{Ext}^1(K-D,D) - \mathbb{P}\ker(\delta) & \xrightarrow{\delta} & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \\
{\scriptstyle \epsilon_D}\downarrow & & \\
\mathcal{W}_D & &
\end{array} \tag{2.4}$$

where $\mathcal{W}_D$ is defined to be $\mathcal{W}_D := \epsilon_D(\mathbb{P}\mathrm{Ext}^1(K-D,D))$. We now define subvarieties of $\mathbb{P}\mathrm{Ext}^1(K-D,D)$:

$\Omega_D^0 = \mathbb{P}\ker(\delta)$,

$\Omega_D^n = \delta^{-1}(\mathrm{Sec}^n(\mathrm{Ver}\mathbb{P}H^1(D))) \cup \mathbb{P}\ker(\delta)$ where $n = 1, \ldots, g-d+r(D)$.

If Ω_D^0 is nonempty we have a sequence of cones:

$$\Omega_D^0 \subset \Omega_D^1 \subset \cdots \subset \Omega_D^{g-d+r(D)} \subset \mathbb{P}\mathrm{Ext}^1(K-D,D),$$

where Ω_D^0 is the vertex.

By considering the long exact sequence in cohomology and the definition of the cones Ω_D^n, the number of sections of an extension is:

$$h^0(C, E) = g + 1 - \text{Cliff}(D) - n \text{ for } E \in \Omega_D^n - \Omega_D^{n-1}. \qquad (2.5)$$

For more details on the construction of the above commutative diagram and the map δ we refer to [23] section 3.

By noting the relationship between Cliff(D) and Cliff(C) (see [23]) the following refinement of equation (2.5) is reached.

Proposition 2.1.1. *If $[E]$ is the equivalence class in $\mathcal{SU}(2, K)$ of the bundle E then we have $h^0(C, E) \leq g + 1 -$ Cliff(C).*

We now give a lemma that relates the maximal subbundles of an extension with the secant variety in which it lies. Suppose that $E \in \mathbb{P}\text{Ext}^1(K - D, D)$ for some D, then the following is Proposition 2.4 of Lange-Narasimhan [13], tailored to our situation.

Proposition 2.1.2. *There is a bijection, given by $L = K(-D - D')$, between:*

1. *line bundles $L \subset E$, $L \neq \mathcal{O}(D)$, of maximal degree; and*

2. *line bundles $\mathcal{O}(D')$of minimal degree such that $E \in \overline{D'}$.*

Here we are using the notation that $\overline{D'}$ is the span of the divisor D'. This notation will be used for the rest of the thesis.

2.2 Existence properties of Brill-Noether loci for $3 \leq g \leq 10$

In this section we discuss the existence property, that $\mathcal{W}^r \neq \emptyset$ for $\rho(r) \geq 0$. We have seen previously that this is an essential consideration when studying the Brill-Noether conditions, as it allows us to impose a lower bound on the dimension of the Brill-Noether loci (see Theorem 1.1.2). In Bertram-Feinberg [4] existence results are stated for genus $3 \leq g \leq 12$ although no details are given for $g \geq 7$. For genus $3 \leq g \leq 6$ Oxbury-Pauly-Previato [23] not only show the existence of these Brill-Noether loci but also give explicit descriptions of the loci with larger numbers of sections. These results are obtained using the material paraphrased in section 2.1. Descriptions of the Brill-Noether loci for genus $7 \leq g \leq 9$ with maximal $\rho(r)$ have been shown by Mukai with an approach using homogeneous spaces. The references for these genera are [19] for genus 7, [16] and [17] for genus 8 and [19] for genus 9.

We now prove that the $\mathcal{W}^r$ are nonempty for $\rho(r) \geq 0$ and $7 \leq g \leq 10$. Furthermore, in the genus 8 case we give a description of $\overline{\mathcal{W}^5}$ for nontetragonal curves. In the following the proofs are given genus by genus and take the approach of [23], drawing heavily on the results about extension spaces.

To prove that $\mathcal{W}^r \neq \emptyset$ for all r such that $\rho(r) \geq 0$ it is sufficient to show that $\mathcal{W}^R \neq \emptyset$ where

$$R := \max\{r \mid \rho(r,d) \geq 0\},$$

as by our definition of the Brill-Noether locus $\mathcal{W}^0 \supseteq \cdots \supseteq \mathcal{W}^R$.

Genus 7

For this genus we study $\mathcal{W}^4$ because 4 is the maximum value of r for which $\rho(r)$ is non-negative. In fact Mukai in [19] proves for a curve C of genus 7 with $W_4^1 = \emptyset$ that $\mathcal{W}^4$ is a Fano 3-fold of Picard number 1 and genus 7. However, we only require that $\mathcal{W}^4 \neq \emptyset$; we verify this by directly constructing bundles of this sort.

Proposition 2.2.1. *For a generic curve C of genus 7 the Brill-Noether locus $\mathcal{W}^4$ is nonempty.*

Proof. We would like to pick a divisor that has among its extensions a stable bundle with 5 sections. Consequently we choose $D \in S^5C$ such that $r(D) = 1$. We know this can be done because $\dim(G_5^1) = \rho(1,5) = 1$. Now we construct the commutative diagram (2.4), which requires that we calculate the dimension of some cohomology groups. By Riemann-Roch $h^1(C,D) = h^0(C,D) - d + g - 1 = 3$, from which we deduce $\dim(\mathrm{Sym}^2 H^1(C,D)) = 6$. A further use of Riemann-Roch tells us that:

$$\dim(\mathrm{Ext}^1(K-D,D)) = h^0(C,2K-2D) = \deg(2K-2D) - g + 1 = 8.$$

Putting this information into the diagram gives:

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & \mathbb{P}H^1(C,D) \cong \mathbb{P}^2 \arrow[d, "\mathrm{Ver}"] \\ \mathbb{P}^7 \cong \mathbb{P}\mathrm{Ext}^1(K-D,D) \arrow[r, "\delta"] & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \cong \mathbb{P}^5. \end{tikzcd}$$

From equation (2.5) we know that if $E \in \Omega_D^n - \Omega_D^{n-1}$ then $h^0(C,E) = 8 - \mathrm{Cliff}(D) - n$. In this extension space E has $h^0(C,E) = 5$ if and only if $E \in \Omega_D^0$. Now we know that Ω_D^0 contains extensions with 5 sections, we would like to show that $\Omega_D^0 \neq \emptyset$ and also that the image of $\epsilon_D : \Omega_D^0 \to \mathcal{SU}(2,K)$ contains stable bundles. From the diagram we see that $\dim(\ker(\delta)) \geq 2$ so $\dim(\Omega_D^0) \geq 1$.

To show that Ω_D^0 contains stable extensions we use Lange-Narasimhan Lemma 2.1.2. The maximal subbundles of E are of the form $K(-D-D')$, where $E \in \overline{D'}$ for some effective divisor D'. Therefore the degree of a maximal subbundle of E is $\deg(K(-D-D') = 2g - 2 - 5 - \deg(D') = 7 - \deg(D')$. The divisor D' is effective so the highest degree of a subbundle will be 6, which means E is at least semistable and may be stable. Hence the semistable extensions are points of the the curve $\lambda_{|2K-2D|}(C)$. The semistable extensions with the correct number of sections will be the intersection points of the curve with Ω_D^0. However, the degree of the projected curve $\delta(\lambda_{|2K-2D|})(C)$ will drop by the number of intersection points of Ω_D^0 and $\lambda_{|2K-2D|}(C)$. Note that $\lambda_{|K-D|}$ is birational because $\deg(K-D) = 7$ and non-degeneracy of $\lambda_{|K-D|}$ precludes the curve being mapped $7:1$ onto a line. The commutative diagram then tells us that $\deg(\delta(\lambda_{|2K-2D|})(C)) = \deg(\mathrm{Ver}(\lambda_{|K-D|}(C))) = 2.7$, which is the degree of the curve in $\mathbb{P}\mathrm{Ext}^1(K-D,D)$. Therefore the curve cannot meet the vertex of the cone, and all extensions with 5 sections are stable. Hence $\mathcal{W}^4$ is nonempty because $\mathcal{W}^4 \supset \mathcal{W}_D^4 = \epsilon_D(\Omega_D^0) \neq \emptyset$. □

Genus 8

For genus 8 the highest value of r such that $\rho(r) \geq 0$ is 5, in which case $\rho(5) = 0$. This suggests that $\mathcal{W}^5$ should be a finite set of bundles. In fact Mukai (see [16] and [17]) has shown that for a generic curve this Brill-Noether locus is a single stable bundle.

For this particular genus we may broaden our area of interest to nontetragonal curves rather than assuming genericity.

Proposition 2.2.2. *Let C be a nontetragonal curve of genus 8. The Brill-Noether locus $\overline{\mathcal{W}^5}$ consists of a single bundle; which is stable if and only if $W^2_7 = \emptyset$.*

Proof. To construct a semistable bundle E with 6 sections we look in extension spaces of suitable divisors. Choose an effective divisor D, from equation (2.5) we have:

$$h^0(C, E) = 9 - \mathrm{Cliff}(D) - n \text{ for } E \in \Omega^n_D - \Omega^{n-1}_D. \tag{2.6}$$

If E is to have 6 sections we require $\mathrm{Cliff}(D) \leq 3$. We will consider extensions of $D \in S^5C$ where $r(D) = 1$, note that divisors of this form exist because $\rho(1, 5) = 0$. To study the extensions of D we use the commutative diagram (2.4). First we calculate the dimensions of the relevant spaces using the Riemann-Roch formula ; $h^1(C, D) = g - d - 1 + h^0(C, D) = 4$ and $\dim(\mathrm{Ext}^1(K - D, D)) = h^0(C, 2K - 2D) = \deg(2K - 2D) - g + 1 = 11$. Plugging this information into our diagram gives:

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & \mathbb{P}H^1(C, D) \cong \mathbb{P}^3 \arrow[d, "\mathrm{Ver}"] \\ \mathbb{P}^{10} \cong \mathbb{P}\mathrm{Ext}^1(K - D, D) \arrow[r, "\delta"] & \mathbb{P}\mathrm{Sym}^2 H^1(C, D) \cong \mathbb{P}^9. \end{tikzcd} \tag{2.7}$$

By equation (2.6) the extensions which have 6 sections lie in $\Omega^0_D = \mathbb{P}\ker(\delta)$. By comparing the dimensions of $\mathrm{Ext}^1(K - D, D)$ and $\mathrm{Sym}^2 H^1(C, D)$, from the above diagram, we have that $\dim(\ker(\delta)) \geq 1$; the space Ω^0_D is non-empty.

We would like to give a more precise dimension count for Ω^0_D. If δ_D is surjective then $\mathbb{P}\ker(\delta) = \{E\}$, for some bundle E. We know that δ is surjective if and only if its dual $\delta^\star : \mathrm{Sym}^2 H^0(C, D) \to H^0(C, 2K - 2D)$ is injective. However, $\delta^\star$ is injective if and only if the image of C under the linear series $|K - D|$ does not lie in any quadric in $\mathbb{P}^3$.

Three preliminary results (Lemmas 2.2.3, 2.2.5 and 2.2.4) on the curve are shown that will enable us to prove in Lemma 2.2.6 that the image of C cannot lie in a quadric.

Lemma 2.2.3. *Let C be a nontrigonal curve of genus 8, and $\mathcal{O}(D) \in W^1_5$. Then C is birational to its image in $\mathbb{P}H^1(C,D)$ under $f : C \xrightarrow{|K-D|} \mathbb{P}H^1(C,D) \cong \mathbb{P}^3$.*

Proof. The map f is given by the linear series $|K - D|$, which has degree 9. Consequently there are 3 possible ways that C can be mapped into $\mathbb{P}^3$: as a $9 : 1$ cover of $\mathbb{P}^1$, a $3 : 1$ cover of a cubic or birationally. The map f is non-degenerate so C cannot be mapped to a linear subspace - this discounts the first possibility. To address the second case we note that a curve of degree n in $\mathbb{P}^n$ is a rational normal curve; so $f(C)$ would be a rational normal cubic curve, causing $f(C)$ to be trigonal. Assuming that C is nontrigonal means that f must be birational. □

Lemma 2.2.4. *Let C be a nontetragonal genus 8 curve, and f a map given by the linear system $|K - D|$ for $\mathcal{O}(D) \in W^1_5$. Then $f(C)$ is smooth or has singularities of multipicity 2.*

Proof. Suppose that $f(C)$ is singular. If we have a singularity of multiplicity $m > 2$ then projecting from this point of the curve will give a map $C \to \mathbb{P}^2$ which is defined by a g^2_{9-m}. The non-degeneracy of f implies that the projection is also non-degenerate. To start with $9 - m > 4$ as otherwise C would be hyperelliptic. Moreover, we cannot have a g^2_5 because the degree-genus formula precludes having a plane quintic of genus 8:

$$g \leq \frac{(5-1)(5-2)}{2} = 6.$$

The only possibility is to have a g^2_6. If $f(C)$ is mapped $3 : 1$ onto a conic then the curve has a trigonal pencil and are finished. Likewise if projection takes $f(C)$ $2 : 1$ onto a singular cubic the curve is hyperelliptic because such a cubic has genus 0 (by degree-genus formula). On the other hand, if the cubic is smooth, then projection from a point of the cubic gives a $4:1$ map from $f(C)$ to $\mathbb{P}^1$ and the curve has a tetragonal pencil. Finally, if the g_6^2 gives a birational map we look at the degree-genus formula again:

$$8 = \frac{(6-1)(6-2)}{2} - B.$$

This implies that a genus 8 planar sextic must have at least one singularity of multiplicity 2 (either two nodes or one tacnode). Projecting from a singularity would gives a g_4^1. This finishes our proof. □

If we limit our interest to generic curves we get a stronger result.

Lemma 2.2.5. *Let C be a generic curve of genus 8, and $\mathcal{O}(D) \in W_5^1$. Then the image of C under $f : C \xrightarrow{|K-D|} \mathbb{P}^3$ is a smooth curve of degree 9.*

Proof. Suppose for a contradiction that $f(C)$ has a singularity of multiplicity m say. Then projecting from the singularity gives a map of the curve into $\mathbb{P}^2$ which is given by g_{9-m}^2. However, for $m \geq 2$ the Brill-Noether number is $\rho(2, 9-m) < 0$. This means that such a linear series cannot exist on a generic curve, and it follows that $f(C)$ must be smooth. □

Lemma 2.2.6. *Suppose C is a nontetragonal curve of genus 8; let $f : C \to \mathbb{P}^3$ be given by the linear series $|K-D|$ where $\mathcal{O}(D) \in W_5^1$. The image $f(C)$ cannot lie on a quadric.*

Proof. There are four different cases to check, corresponding to the rank of the quadric.

Assume first that the rank of the quadric is 1 or 2; so C would be mapped to a double plane or a pair of planes. The curve is connected so it must lie in just one plane which contradicts the non-degeneracy of f.

By Lemma 2.2.3 f maps C birationally onto a curve of degree 9 and genus 8 in $\mathbb{P}^3$; by abuse of notation C will denote both C and $f(C)$.

A quadric of rank 3 is the quadric cone, which we denote by X. Let π be the projection down the cone onto the base conic. The degree of C is 9, so for such a projection to occur the curve must pass through the vertex of X an odd number of times. Assuming that C passes through the vertex $2r+1$ times (here $0 \le r \le 4$) then the projection π will map the curve $d:1$ onto the base conic, where $d = \frac{1}{2}(9-(2r+1)) = 4-r$. The linear series on C associated to the map $\pi|_C$ will be a g^1_d. However, we assumed C to be non-tetragonal so we have obtained a contradiction.

We now move on to the case in which the quadric is smooth. A rank 4 quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Now pick E_0 and B lines lying in opposite rulings; together they generate the homology of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $C \sim m_1 E_0 + m_2 B$ where $m_1 + m_2 = \deg(C) = 9$, so at least one of m_1 and m_2 is less than 5. Projection from the generator with the larger intersection with C is an $m_i : 1$ map to $\mathbb{P}^1$ where $m_i < 5$. This contradicts the fact that C is nontetragonal.

□

We have shown that Ω^0_D consists of a single extension E; but now we must consider the stability of this bundle. We show that E must be semistable by using Lange-Narasimhan Lemma 2.1.2. A maximal subbundle of E is $K(-D-D')$, where $E \in \overline{D'}$ for some effective divisor D'. The highest degree a subbundle can take is therefore 8, which corresponds to E being unstable and lying in $\Omega^0_D \cap \lambda_{|2K-2D|}(C)$. If this were the case, then $\deg(\delta(\lambda_{|2K-2D|}(C))) = \deg(\lambda_{|2K-2D|}(C)) - 1$. However, the commutativity of our diagram and birationality of $\lambda_{|K-D|}$ tells us that the degree of C in $\mathbb{P}\mathrm{Sym}^2 H^1(C,D)$ is $\deg(\mathrm{Ver}(\lambda_{|K-D|}(C))) = \deg(\lambda_{|2K-2D|}(C))$. Hence E must be at least a semistable extension and may be stable.

We have shown that each $\mathcal{O}(D) \in W^1_5$ gives rise to a single bundle in $\mathcal{W}^5$, or using our notation $\mathcal{W}^5_D = \{E\}$. It is not clear though what the intersection of the two spaces $\mathcal{W}^5_D$ and $\mathcal{W}^5_{D'}$ is, for $\mathcal{O}(D), \mathcal{O}(D') \in W^1_5$. It will now be shown that in fact $\mathcal{W}^5 = \mathcal{W}^5_D$ for all $D \in S^5C$ with $r(D) = 1$. To do this we use a result due to Mukai [17] Proposition 3.1.

Lemma 2.2.7. *If $|D|$ is basepoint free then for any rank two vector bundle F with canonical determinant we have:*

$$h^0(C, F(-D)) \geq h^0(C, F) - deg(D).$$

This result will be used by noting that $H^0(C, E(-D)) = \mathrm{Hom}(\mathcal{O}(D), E)$. Choose a particular $D \in S^5C$ such that $r(D) = 1$, then for any $E \in \mathcal{W}^5$ we have:

$$h^0(C, E(-D)) \geq h^0(C, E) - \deg(D) = 6 - 5 = 1. \tag{2.8}$$

Hence for all $E \in \mathcal{W}^5$ there is a homomorphism $h_E : \mathcal{O}(D) \to L \subset E$, for some line subbundle L. The subbundle $L = \mathcal{O}(D')$ for a divisor D', where $D \subset D'$. Recall equation (2.6), which stated that if E has 6 sections then $\mathrm{Cliff}(L) \leq 3$. However, E is semistable so $\deg(L) \leq 7$. We can discount the possibility that $L \in W^2_6$ because the existence of such a bundle implies that the curve is tetragonal (see proof of Lemma 2.2.4). We must have either $L \in W^1_5$ or $L \in W^2_7$.

This brings us to the point where we consider whether W^2_7 is empty or not.

Assuming that $W^2_7 = \emptyset$ means that we cannot have any semistable bundles with 6 sections which are not stable. That is to say there is no proper subbundle $F \subset E$ such that $\mu(F) = \mu(E)$. For a contradiction suppose that

$$0 \to \mathcal{O}(D_0) \to E \to K(-D_0) \to 0$$

for $D_0 \in S^7C$. Noting by Riemann-Roch that $h^0(C, D_0) = h^1(C, D_0)$, then $6 = h^0(C, E) \leq h^0(C, D_0) + h^1(C, D_0) = 2h^0(C, D_0)$. This gives us $\mathcal{O}(D_0) \in W^2_7$.

The emptiness of W^2_7 also tells us that the image of $\mathcal{O}(D)$ under h_E is $\mathcal{O}(D)$ itself, therefore:

$$h_E : \mathcal{O}(D) \hookrightarrow E \text{ for all } E \in \mathcal{W}^5 \Rightarrow \mathcal{W}^5 = \mathcal{W}^5_D. \tag{2.9}$$

We conclude that $\mathcal{W}^5 = \{E\}$, a single stable bundle.

Now assume that $W^2_7 \neq \emptyset$. Let $\mathcal{O}(D') \in W^2_7$, then $\mathcal{O}(D') \oplus K(-D')$ is a semistable bundle with 6 sections. By Mukai's Lemma 2.2.7, we have that for any of our D

above a homomorphism $h_D : \mathcal{O}(D) \to L \subset \mathcal{O}(D') \oplus K(-D')$. The map h_D may drop rank at points of the curve so $\deg(L) \geq \deg(D) = 5$.

A bundle $E \in \overline{\mathcal{W}^5}$ is an extension of a bundle in W^1_5 or W^2_7; so there are only three possibilities for the bundle L, it is either $\mathcal{O}(D)$ itself or one of $\mathcal{O}(D')$ and $K(-D')$. In the first instance, we have that $\Omega^0_D = \{\mathcal{O}(D') \oplus K(-D')\}$ (note that Ω^0_D is a single point by Lemma 2.2.6), and so $\mathcal{W}^5_D = \{\mathcal{O}(D') \oplus K(-D')\}$.

In the second case, both possibilities are equivalent so we assume that $L = \mathcal{O}(D')$. To obtain the result $\mathcal{W}_D = \{\mathcal{O}(D') \oplus K(-D')\}$ we show that L is a subbundle of E, where $\{E\} = \mathcal{W}^5_D$. The map h_D is zero on the fibres over the support of D and moreover drops rank at two further points so it must be the case that $D \subset D'$.

We now consider the image of the curve under $f : C \xrightarrow{|K-D|} \mathbb{P}H^1(C, D)$. The map f can be seen as the projection of the canonical curve away from the span of D. The geometric Riemann-Roch formula gives us the following information about the dimensions of the spans of D and D' in canonical space:

$$\begin{aligned} \dim(\overline{D}) &= (5-1) - 1 = 3, \\ \dim(\overline{D'}) &= (7-1) - 2 = 4. \end{aligned} \tag{2.10}$$

We know that $D \subset D'$, so equations (2.10) tell us that on projection away from $\overline{D}$ the span of D' is mapped to a point. In particular two points p and q on $C \subset \mathbb{P}^7$ given by $p + q = D' - D$ are mapped to a point in $\mathbb{P}H^1(D)$ and the curve possesses a singularity of multiplicity 2 at $f(p) = f(q)$. The image of the curve under the Veronese embedding will also have a singularity of the same multiplicity. However, the curve in $\mathbb{P}\mathrm{Ext}^1(K - D, D)$ is smooth (because $\deg(2K - 2D) = 18 = 2g + 2$ and so has sufficiently high degree to be an embedding). Hence for the curve to pick up a singularity in $\mathbb{P}S^2H^1(C, D)$ the two points p and q in the extension space must be mapped to one by δ. The map δ is projection away from the point vertex Ω^0_D, which therefore must lie on the 2-secant line $\overline{pq}$.

We now use Lange-Narasimhan's result (see Lemma 2.1.2) that maximal subbundles of E are of the form $K(-D-D_0)$ where $E \in \overline{D_0}$. In our case we know that E is semistable so in particular $K(-D-p), K(-D-q) \not\subseteq E$. Therefore $K(-D') = K(-D-p-q) \subset E$. Again we have that $\mathcal{W}^5_D = \{\mathcal{O}(D') \oplus K(-D')\}$.

We have shown that for all $\mathcal{O}(D) \in W^1_5$, $\mathcal{W}^5_D = \{\mathcal{O}(D') \oplus K(-D')\}$, for a single $\mathcal{O}(D') \in W^2_7$. On the other hand the arguments above give us that for all $L \in W^2_7$, $\{L \oplus KL^{-1}\} = \mathcal{W}^5_D$ for a single $D \in S^5C$ with $r(D) = 1$. The bundles $E \in \overline{\mathcal{W}^5}$ are extensions of either $\mathcal{O}(D) \in W^1_5$ or $L \in W^2_7$ which have been shown to be of the form $\mathcal{O}(D') \oplus K(-D')$ in either case. Thus we conclude that if $W^2_7 \neq \emptyset$ then $\overline{\mathcal{W}^5}$ consists of a single semistable bundle. □

We can now state an easy corollary.

Corollary 2.2.8. *Let C be a generic curve of genus 8, then $\mathcal{W}^5$ is a single stable bundle.*

Proof. By Propostion 2.2.2 it is enough to show that a generic curve has no g^2_7s. The Brill-Noether number is $\rho(2,7) = -2 < 0$, so for a generic curve $W^2_7 = \emptyset$. □

Genus 9

For this genus we study $\mathcal{W}^5$. Mukai [19] stated (without proof) that this locus is a singular quartic threefold. Here we show an existence result; for a fuller description of $\mathcal{W}^5$ see chapter 5.

Proposition 2.2.9. *For a generic curve C of genus 9 the Brill-Noether locus $\mathcal{W}^5$ is nonempty.*

Proof. The Brill-Noether theory of line bundles tells us that there exist $D \in S^6C$ such that $r(D) = 1$, because $\rho(1,6) = 1$. The extension space $\mathbb{P}\mathrm{Ext}^1(K-D, D)$ associated to this divisor is now studied in order to find a stable bundle with the right number of sections. The formula (2.5) when applied in this case gives:

$$E \in \Omega^n_D - \Omega^{n-1}_D \Rightarrow h^0(C,E) = 10 - \mathrm{Cliff}(D) - n.$$

However, Cliff(D) $=\deg(D) - 2.r(D) = 6 - 2.1 = 4$, so we are interested in the subspace $\Omega_D^0 \subset \mathbb{P}\mathrm{Ext}^1(K-D, D)$.

By Riemann-Roch we calculate $h^0(C, 2K-2D) = \dim(\mathbb{P}\mathrm{Ext}^1(K-D,D)$ and $h^1(C,D)$ and plug these into commutative diagram (2.4) to give us:

$$\begin{array}{ccc} C & \xrightarrow{|K-D|} & \mathbb{P}H^1(C,D) \cong \mathbb{P}^3 \\ {\scriptstyle |2K-2D|}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}} \\ \mathbb{P}^{11} \cong \mathbb{P}\mathrm{Ext}^1(K-D,D) & \xrightarrow{\delta} & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \cong \mathbb{P}^9. \end{array}$$

From looking at the diagram we observe that $\dim(\ker(\delta)) \geq 2$, therefore $\Omega_D^0 \cong \mathbb{P}^n$ for $n \geq 1$. In order to show that Ω_D^0 contains some stable bundles we require that $\lambda_{|K-D|}$ is birational.

Lemma 2.2.10. *Let C be a generic curve of genus 9 and $\lambda : C \overset{|K-D|}{\longrightarrow} \mathbb{P}H^1(C,D) \cong \mathbb{P}^3$, where $D \in S^6C$ with $r(D) = 1$. The map λ is birational.*

Proof. The degree of λ is $\deg(K-D) = 2g-2-6 = 10$. Hence λ can map C into $\mathbb{P}^3$ in one of four ways, as a $10:1$ cover of a line, a $5:1$ cover of a conic, a $2:1$ cover of a quintic or birationally. The first two possibilities constrain the curve $\lambda(C)$ to lie in a line or plane respectively; which contradicts the nondegeneracy of λ. Now consider the possibility that λ maps the curve $2:1$ onto a quintic C_0. However, the involutive sheet interchange map associated to the map $\lambda : C \overset{2:1}{\to} C_0$ is an automorphism of C. Generic curves of genus at least three have no automorphisms (see Griffiths-Harris [11] page 276). We conclude that λ must be birational. □

The Lange-Narasimhan Lemma 2.1.2 states that the maximal subbundles of the extension E are of the form $K(-D-D')$ where $E \in \overline{D'}$ for some effective divisor D'. The degree of a maximal subbundle must therefore be $2g-2-6-\deg(D') = 10 - \deg(D')$. The highest degree that a subbundle can have is 9, which means that the extension is unstable and lies on the curve $\lambda_{|2K-2D|}(C)$. If $\deg(D') = 2$ the extension is semistable and lies on a 2-secant of the the curve. For D' with higher degree the extensions are stable.

An unstable extension is a point of the intersection $\Omega_D^0 \cap \lambda_{|2K-2D|}(C)$. On projection down the cone to $\mathbb{P}\mathrm{Sym}^2 H^1(C, D)$ the degree of the curve drops. However, by our proof that $\lambda_{|K-D|}$ is birational and the commutativity of the diagram we know that the degree of the curve in $\mathbb{P}\mathrm{Ext}^1(K-D, D)$ is equal to the degree of the curve in $\mathbb{P}\mathrm{Sym}^2 H^1(C, D)$. We must have that all extensions in Ω_D^0 are semistable, some of which may be stable.

We now show that some of the extensions are stable. For a contradiction suppose that all E in the vertex Ω_D^0 are semistable. This condition holds if and only if $\Omega_D^0 \subset \mathrm{Sec}^2(C)$. If every point of the vertex lies on a 2-secant then projecting down the cone will map the curve $2:1$ onto a curve in $\mathbb{P}\mathrm{Sym}^2 H^1(C, D)$. However, the map from $\mathbb{P}H^1(C, D)$ to $\mathbb{P}\mathrm{Sym}^2 H^1(C, D)$ is an embedding which means $\lambda_{|K-D|}$ maps C $2:1$ onto a curve in $\mathbb{P}H^1(C, D)$. This is a contradiction because $\lambda_{|K-D|}$ is birational. Therefore we must have stable extensions in Ω_D^0. We conclude that $\mathcal{W}^5 \neq \emptyset$ since $\overline{\mathcal{W}^5} \supset \mathcal{W}_D^5 = \epsilon_D(\Omega_D^0)$ and $\epsilon_D(\Omega_D^0)$ contains stable bundles. □

Genus 10

First we note that 5 is the highest r such that $\rho(r) \geq 0$, so we now show the following result.

Proposition 2.2.11. *For C a generic curve of genus 10 the Brill-Noether locus $\mathcal{W}^5$ is nonempty.*

Proof. Choose a divisor $D \in S^7C$ with $r(D) = 1$; this is possible because $\dim(G_7^1) = \rho(1, 7) = 2$. Equation (2.5) gives us that:

$$E \in \Omega_D^n - \Omega_D^{n-1} \Rightarrow h^0(C, E) = 11 - \mathrm{Cliff}(D) - n = 6 - n.$$

Hence the vertex Ω_D^0 contains the extensions with 6 sections. To obtain our existence result we first construct the commutative diagram (2.4). By Riemann-Roch $h^1(C, D) = 4$ and $\dim(\mathrm{Sym}^2 H^1(C, D)) = 10$, moreover $\dim(\mathrm{Ext}^1(K-D, D)) =$

$h^0(C, 2K-2D) = 13$, also by Riemann-Roch. We then have the following diagram.

$$\begin{array}{ccc} C & \xrightarrow{|K-D|} & \mathbb{P}H^1(C,D) \cong \mathbb{P}^3 \\ {\scriptstyle |2K-2D|}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}} \\ \mathbb{P}^{12} \cong \mathbb{P}\mathrm{Ext}^1(K-D,D) & \xrightarrow{\delta} & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \cong \mathbb{P}^9. \end{array}$$

We observe that $\dim(\ker(\delta)) = 13 - \mathrm{rank}(\delta) \geq 13 - 10 = 3$, so $\dim(\Omega_D^0) \geq 2$. We now show that some of the extensions lying in Ω_D^0 must be stable. Again by Lange-Narasimhan 2.1.2 the maximal subbundles of the extension E are of the form $K(-D-D')$ for $E \in \overline{D'}$ for some effective D'. The degree of a maximal subbundle must therefore be $2g-2-7-\deg(D') = 11 - \deg(D')$. If $\deg(D') = 1$ then the extension is unstable and lies on the curve. Otherwise the extension is at least semistable and may be stable.

To show that Ω_D^0 contains stable extensions we require that $\lambda_{|K-D|}$ is birational. This is clear though as $\deg(K-D) = 11$ and we cannot have C being mapped $11:1$ onto a line as this contradicts the non-degeneracy of $\lambda_{|K-D|}$.

We can dispense with the unstable extensions by noting that if $\Omega_D^0 \cap \lambda_{|2K-2D|}(C) \neq \emptyset$ then the degree of the curve will drop when we project to $\mathbb{P}\mathrm{Sym}^2 H^1(C,D)$; this contradicts the commutativity of our diagram which forces $\lambda_{|2K-2D|}(C)$ and the projected curve to have the same degree.

To show that some of the extensions are stable suppose for a contradiction that all $E \in \Omega_D^0$ are semistable and not stable. We must have $\Omega_D^0 \subset \mathrm{Sec}^2(C)$, so projection down the cone maps $\lambda_{|2K-2D|}(C)$ $2:1$ onto a curve in $\mathbb{P}\mathrm{Sym}^2 H^1(C,D)$. The embedding $\mathrm{Ver}: \mathbb{P}H^1(C,D) \to \mathbb{P}\mathrm{Sym}^2 H^1(C,D)$ forces $\lambda_{|K-D|}$ to be $2:1$, which contradicts birationality.

The image of Ω_D^0 under the moduli map ϵ_D is nonempty and lies in $\mathcal{W}^5 \subset \mathcal{SU}(2,K)$, so $\mathcal{W}^5 \neq \emptyset$. $\square$

2.3 Nonexistence properties

In this section we consider the nonexistence Brill-Noether condition (1.9). Recall this asserts that for a generic curve with $\rho(r) < 0$ then $\mathcal{W}^r = \emptyset$. Bertram-Feinberg [4] prove this result for genus $3 \leq g \leq 6$ and state it for $7 \leq g \leq 9$ and $g = 11, 13$. Our aim is to prove the following proposition.

Proposition 1.3.6 *Let C be a generic curve of genus $g \leq 11$ or $g = 13$. If $\rho(r) < 0$, then $\mathcal{W}^r = \emptyset$.*

In Proposition 2.3.1 we prove the nonexistence condition for genus $3 \leq g \leq 9$ and $g = 11, 13$ by using the Clifford index of the curve. We deal with the genus 10 case separately in Proposition 2.3.2; the proof of which uses a remark of Mukai [15].

Proposition 2.3.1. *Let C be a generic curve of genus $3 \leq g \leq 9$ or $g = 11, 13$. If $\rho(r) < 0$ then $\mathcal{W}^r = \emptyset$.*

Proof. To prove this result we will use inequality in Proposition 2.1.1:

$$h^0(C, E) \leq g + 1 - \mathrm{Cliff}(C).$$

However, C generic implies $\mathrm{Cliff}(C) = [\frac{g-1}{2}]$ (see equation (2.2)). From now on it will be necessary to consider whether g is odd or even.

First let us assume that $g = 2k + 1$. Then inequality (2.2) becomes:

$$h^0(C, E) \leq (2k + 1) + 1 - \left[\frac{(2k + 1) - 1}{2}\right] = k + 2.$$

Now take the maximum value and substitute into the Brill-Noether number:

$$\begin{aligned}
\rho(k + 1) &= 3g - 3 - \frac{1}{2}(k + 2)(k + 3) \\
&= 3(2k + 1) - 3 - \frac{1}{2}(k + 2)(k + 3) \\
&= \frac{7}{2}k - \frac{1}{2}k^2 - 3 \\
&= \frac{1}{2}(6 - k)(k - 1).
\end{aligned}$$

From this calculation we can see that $\rho(r)$ will become negative when $k \geq 7$ (note that at the other extreme we are not interested in elliptic curves). Therefore, if $k \leq 6$ and $\mathcal{W}^r \neq \emptyset$ (note $r \leq k+1$) then we must have $\rho(r) \geq \rho(k+1) \geq 0$. Hence generic curves of odd genus $g \leq 13$ with $\rho(r) < 0$ have $\mathcal{W}^r = \emptyset$.

Now consider the case when $g = 2k$. Our upper bound on $h^0(C, E)$ becomes:

$$h^0(C,E) \leq (2k) + 1 - \left\lceil \frac{2k-1}{2} \right\rceil = k+2$$

Again substituting into the Brill-Noether number gives:

$$\begin{aligned} \rho(k+1) &= 3g - 3 - \frac{1}{2}(k+2)(k+3) \\ &= 3(2k) - 3 - \frac{1}{2}(k+2)(k+3) \\ &= \frac{7}{2}k - \frac{1}{2}k^2 - 6 \\ &= \frac{1}{2}(4-k)(k-3). \end{aligned}$$

This tells us that generic curves of even genus can have $\rho(r) < 0$ and line bundles of the correct Clifford index for genus $g = 4$ or $g \geq 10$. Although on the face of it this does not seem promising it is possible that these line bundles have degree $g-1$ and so give rise to semistable bundles that do not lie in the stable locus $\mathcal{W}^r$. When $g = 4$ we show that this is in fact the case by using Mukai's Lemma (see (2.2.7)). This states that for a basepoint free linear system $|D|$ we have

$$h^0(C, E(-D)) \geq h^0(C, E) - \deg(D).$$

For $g = 4$ we are looking at $E \in \mathcal{W}^3$, so for $\mathcal{O}(D) \in W^1_3$ we have $h^0(C, E(-D)) \geq 4 - 3 = 1$. Hence E is semistable but not stable. Thus verifying the condition that if C is generic of even genus $g < 10$ and $\rho(r) < 0$, then $\mathcal{W}^r = \emptyset$. □

We now consider the question of nonexistence for genus 10 generic curves. Noting that the maximal value of r for $\rho(r) \geq 0$ is $r = 5$ it is enough to show that $\mathcal{W}^6 = \emptyset$.

Proposition 2.3.2. *Let C be a generic curve of genus 10, then $\overline{\mathcal{W}^6} = \emptyset$.*

Proof. For a contradiction suppose that we have a semistable bundle E with $h^0(C, E) = 7$. We start by choosing a divisor D on C that has generic Clifford index so that its extension space $\mathbb{P}\mathrm{Ext}^1(K - D, D)$ will contain bundles with 7 sections and degree low enough so that we may use Mukai's lemma 2.2.7.

Take $D \in S^6C$ with $r(D) = 1$; we know that such divisors exist on a generic curve because $\rho(1,6) = 0$ (in fact we may use Castelnuovo's Theorem 1.1.9 to see that there are exactly 42 such linear series). We have $\mathrm{Cliff}(D) = 6 - 2.1 = 4$, so by equation (2.5) for $E \in \Omega_D^0$, $h^0(C, E) = g + 1 - \mathrm{Cliff}(D) = 7$.

Now Mukai's lemma 2.2.7 tells us that:

$$h^0(C, E(-D)) \geq h^0(C, E) - \deg(D) = 1.$$

We know that $H^0(C, E \otimes \mathcal{O}(-D)) \cong \mathrm{Hom}(\mathcal{O}(D), E)$ so there is a homomorphism $h_E : \mathcal{O}(D) \to L \subset E$. We would like to show that h_E is an injection, so that E is an extension of $\mathcal{O}(D)$. We know that L is a subbundle of E so we have a sequence:

$$0 \to L \to E \to KL^{-1} \to 0.$$

A portion of the associated cohomology sequence is:

$$0 \to H^0(C, L) \to H^0(C, E) \to H^0(C, KL^{-1}) \to \ldots.$$

From this we get the inequality:

$$7 = h^0(C, E) \leq h^0(C, L) + h^0(C, KL^{-1}). \tag{2.11}$$

Now suppose that $\deg(L) = d$, the Riemann-Roch formula tells us:

$$d - 9 = h^0(C, L) - h^0(C, KL^{-1}). \tag{2.12}$$

Now add equations (2.11) and (2.12) to get $d-2 \leq 2h^0(C, L)$, and so $\frac{1}{2}(d-4) \leq r(L)$.

With this inequality we may look at the Brill-Noether number of L:

$$\begin{aligned}\rho(L) \leq \rho(\tfrac{1}{2}(d-4), d) &= 10 - (\frac{d-2}{2})(10 - d + \frac{d-4}{2}) \\ &= 10 + \frac{(d-2)(d-16)}{4} \\ &= \frac{1}{4}(d^2 - 18d + 72) \\ &= \frac{1}{4}(d-6)(d-12).\end{aligned}$$

The curve is generic so $\rho(L) \geq 0$, which means that $d \leq 6$ or $d \geq 12$. However, as E is a semistable bundle the degree of L is at most 9, we must have $d = 6$. Therefore $\mathcal{O}(D) \overset{h_E}{\hookrightarrow} L \subset E$, and E is an extension of $\mathcal{O}(D)$. This is true for any $E \in \overline{\mathcal{W}^6}$ so we must have $\overline{\mathcal{W}^6} = \mathcal{W}^6_D = \epsilon_D \Omega^0_D$.

Moreover, by the above E has no subbundles of degree 9, so E must be stable. This simplifies the result we are trying to prove because $\overline{\mathcal{W}^6} = \mathcal{W}^6$.

We now construct the commutative diagram (2.4) in order to determine the dimension of Ω^0_D. Now $h^1(C, D) = 5$, so $\dim(\mathrm{Sym}^2 H^1(C, D)) = 15$ and $\dim(\mathrm{Ext}^1(K - D, D)) = h^0(2K - 2D) = 15$ by Riemann-Roch. Then:

$$\begin{array}{ccc} C & \xrightarrow{|K-D|} & \mathbb{P}H^1(C,D) \cong \mathbb{P}^4 \\ {\scriptstyle |2K-2D|}\big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \mathrm{Ver}} \\ \mathbb{P}^{14} \cong \mathbb{P}\mathrm{Ext}^1(K-D, D) & \xrightarrow{\delta} & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \cong \mathbb{P}^{14} \end{array}$$

Recall that Ω^0_D is defined to be $\mathbb{P}\ker(\delta)$. If δ is surjective then the kernel of δ is trivial and so $\Omega^0_D = \emptyset$, in which case $\mathcal{W}^6 = \mathcal{W}^6_D = \epsilon_D(\Omega^0_D) = \emptyset$. The map δ is surjective if and only if the image of C under $C \xrightarrow{|K-D|} \mathbb{P}^4$ does not lie in a quadric. In Mukai [15] the following remark is stated.

Remark 0.8 *If C is the generic curve of genus* 10, *then the image $C_{12} \subset \mathbb{P}^4$ embedded by any g^4_{12} is not contained in a quadratic hypersurface.*

This concludes the proof. □

Chapter 3

The Petri map

We have already seen in chapter 1 that the injectivity of the Petri map

$$\mu : \mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E),$$

is central to the conjectures about the Brill-Noether loci. In this chapter we follow the approach of Bertram-Feinberg [4] to the injectivity question. In section 3.1 an equivalent geometric condition is given for the Petri map to be injective. In section 3.2 we use this description to give the strategy that will be adopted in proving injectivity in the following 3 sections.

3.1 Geometric description of injectivity

We start this section by verifying directly that μ is injective for E with $h^0(C,E) \leq 2$.

Lemma 3.1.1. *Let C be an algebraic curve, and $E \in \mathcal{SU}(2,K)$ a stable bundle with $h^0(C,E) = 1,2$. The Petri map $\mu : Sym^2 H^0(C,E) \to H^0(C, Sym^2 E)$ is injective.*

Proof. The first case is clear; suppose that $H^0(C,E) = \langle s \rangle$, then $\mathrm{Sym}^2 H^0(C,E) = \langle s \otimes s \rangle$. Consequently if μ were not injective then;

$$0 = \mu(s \otimes s) = s^2 \Rightarrow s = 0$$

this is a contradiction.

Now look at the case $h^0(C,E) = 2$. Suppose that $H^0(C,E) = \langle s_1, s_2\rangle$, then $\mathrm{Sym}^2 H^0(C,E) = \langle s_1 \otimes s_1, s_1 \otimes s_2, s_2 \otimes s_2\rangle$. Assume that $\mu(\sigma) = 0$ for some

$$\sigma = \alpha s_1 \otimes s_1 + \beta s_1 \otimes s_2 + \gamma s_2 \otimes s_2 \in \mathrm{Sym}^2 H^0(C,E),$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. The homogeneous polynomial $\mu(\sigma) = \alpha s_1^2 + \beta s_1 s_2 + \gamma s_2^2$ may be written as

$$(as_1 - bs_2)(cs_1 - ds_2) = acs_1^2 - s_1 s_2(ad + bc) + bds_2^2 = \alpha s_1^2 + \beta s_1 s_2 + \gamma s_2^2;$$

for some $a, b, c, d \in \mathbb{C}$. This gives us that:

$$0 = \mu(\sigma) = (as_1 - bs_2)(cs_1 - ds_2) \Rightarrow (as_1 - bs_2) \text{ or } (cs_1 - ds_2) = 0.$$

Suppose (without loss of generality) $as_1 = bs_2$, then it must be the case that $a = b = 0$ as otherwise $\{s_1, s_2\}$ would be linearly dependent. It follows that $\alpha = \beta = \gamma = 0$ and so $\sigma = 0$; showing μ to be injective. □

From now on when considering the injectivity of μ at E we will assume that $h^0(C,E) > 2$.

Given a stable bundle $E \in \mathcal{SU}(2,K)$ we use E to define a map

$$\Phi_E : C \to Gr(2, H^0(C,E)^\star),$$

in an equivalent way to line bundles giving maps to projective space. If we identify the 2-planes in $H^0(C,E)^\star$ with lines in $\mathbb{P}H^0(C,E)^\star$ then it transpires that the injectivity of the Petri map μ corresponds to the image of the curve under Φ_E not lying in the lines in any quadric $q \subset \mathbb{P}H^0(C,E)^\star$ (see Lemma 3.1.4). To define the map Φ_E we require that every point of the curve is taken to a line in $\mathbb{P}H^0(C,E)^\star$; this will follow if the bundle E is generically globally generated (see Definition 3.1.2). To tackle the outstanding case Lemma 3.1.3 shows that generic curves of genus $g \leq 11$ cannot possess bundles which are not globally generically generated.

Definition 3.1.2. *The vector bundle E is generically globally generated if the evaluation map:*

$$\epsilon : H^0(C, E) \otimes \mathcal{O} \to E \text{ given pointwise by } \epsilon_p : s \mapsto s(p) \text{ for all } p \in C$$

is surjective for all but a finite number of $p \in C$.

The case of the bundle $E \in \mathcal{W}^r$ not being generically generated is discussed in the following proposition, which is stated without proof in Bertram-Feinberg [4].

Proposition 3.1.3. *Let C be a generic curve of genus $g \leq 11$. Then for all stable $E \in \mathcal{SU}(2, K)$, E is generically generated.*

Proof. Supposing that E is not generically generated then there is a line bundle $L \subset E$ such that $\deg(L) < g - 1$ and $H^0(C, L) = H^0(C, E)$.

We note here that in the case $h^0(C, E) \geq 3$, we must have a line bundle L with a relatively large number of sections and low degree. We consequently look at the Brill-Noether number of a bundle with 3 sections (the minimal number) and degree $g - 2$ (the maximal degree):

$$\rho(2, g-2) = g - (3)(g - (g-2) + 2) = g - 12.$$

For a generic curve this number will be non-negative. Therefore, if $g \leq 11$ the curve cannot have line bundles of this sort and so E must be generically globally generated. □

From now on we can assume that the evaluation map $\epsilon : H^0(C, E) \otimes \mathcal{O} \to E$ is generically surjective. We define divisors on C for which the evaluation map is not surjective:

$$D_1 := \{p \in C \mid \operatorname{im}(\epsilon|_p) = \mathbb{C}\}, \tag{3.1}$$

$$D_2 := \{p \in C \mid \operatorname{im}(\epsilon|_p) = 0\}. \tag{3.2}$$

We now use ϵ to induce a morphism:

$$\Phi_E : C \to Gr(2, H^0(C,E)^\star) \text{ given by } p \mapsto H^0(C, E(-p))^\perp \subseteq H^0(C,E)^\star. \quad (3.3)$$

The map Φ_E is well defined for generically globally generated E because we know that $\dim(H^0(C, E(-p))^\perp) = 2$ for almost all $p \in C$. For $p \in D_1+D_2$ we are implicitly taking the limit of spaces $H^0(C, E(-p_i))^\perp$ for a sequence of points $\{p_i\}_{i=1}^\infty \subset C - \{D_1 + D_2\}$ converging to p. To explain the statement of the following lemma we introduce the notation that for a quadric $q \subset \mathbb{P}H^0(C,E)^\star$ we define $\mathbb{G}r(1,q)$ to be the space of projective lines lying in q.

Lemma 3.1.4. *Let $E \in \mathcal{SU}(2,K)$ be a generically generated stable bundle over a generic curve C. The Petri map at E is injective if and only if $\Phi_E(C)$ is not contained in $\mathbb{G}r(1,q)$ for any quadric q in $\mathbb{P}H^0(C,E)^\star$.*

Proof. Let $\Sigma := \mathbb{P}E^\star$ and consider the line bundle $\mathcal{O}_\Sigma(1)$ defined on the ruled surface $\Sigma = \mathbb{P}E^\star \xrightarrow{\pi} C$. Then

$$\pi_\star(\mathcal{O}_\Sigma(n)) = \mathrm{Sym}^n E, \quad (3.4)$$

moreover $R^i\pi_\star\mathcal{O}(n) = 0$ for $i \neq 0,1$ and for all n (see Hartshorne [12] Exercise 8.4 page 253). For a description of the sections of $\mathrm{Sym}^2 E$ we refer to [12] Exercise 8.1.

Let $f : X \to Y$ is a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X and assume that $R^i f_\star(\mathcal{F}) = 0$ for all $i > 0$; then $H^i(X,\mathcal{F}) \cong H^i(Y, f_\star\mathcal{F})$.

Taking $f = \pi$, $X = \Sigma$, $Y = C$ and $\mathcal{F} = \mathcal{O}_\Sigma(1)$; then

$$H^0(\Sigma, \mathcal{O}(n)) \cong H^0(C, \mathrm{Sym}^n E), \quad (3.5)$$

since all the higher direct image sheaves are zero. We can now construct a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^2 H^0(C,E) & \xrightarrow{\mu} & H^0(C, \mathrm{Sym}^2 E) \\ {\scriptstyle \mathrm{iso}}\downarrow & & \downarrow{\scriptstyle \mathrm{iso}} \\ \mathrm{Sym}^2 H^0(\Sigma, \mathcal{O}(1)) & \xrightarrow{\nu} & H^0(\Sigma, \mathcal{O}(2)). \end{array} \quad (3.6)$$

Using the line bundle $\mathcal{O}_\Sigma(1)$ the ruled surface may be mapped to projective space; this map is denoted by $\alpha : \Sigma \xrightarrow{|\mathcal{O}(1)|} \mathbb{P}H^0(\Sigma, \mathcal{O}(1))^\star \cong \mathbb{P}H^0(C,E)^\star$. The kernel of the multiplication map ν will be those quadrics in $\mathbb{P}H^0(\Sigma, \mathcal{O}(1))^\star$ containing the image of the ruled surface, $\alpha(\Sigma)$.

Suppose that $\mu(\phi) = 0$ for some $\phi \in \mathrm{Sym}^2 H^0(C, E)$. By suppressing the isomorphism $\mathrm{Sym}^2 H^0(C,E) \xrightarrow{\sim} \mathrm{Sym}^2 H^0(\Sigma, \mathcal{O}(1))$ we have $\phi \in \mathrm{Sym}^2 H^0(\Sigma, \mathcal{O}(1))$ and $\nu(\phi) = 0$. Which tells us that $\mu(\phi) = 0$ if and only if the quadric corresponding to ϕ in $\mathbb{P}H^0(\Sigma, \mathcal{O}(1))^\star$ contains $\alpha(\Sigma)$. Consequently μ is injective if and only if $\alpha(\Sigma)$ does not lie in any quadric.

The second stage of the argument is to consider the kernel of μ in terms of the Grassmannian $Gr(2, H^0(C, E)^\star)$. Now consider the diagram:

$$\begin{tikzcd} E^\star \arrow[r, "\beta"] \arrow[d, "\pi"'] & U \arrow[r, "i"] \arrow[d, "\pi"] & H^0(C,E)^\star \otimes \mathcal{O}_G \\ C \arrow[r, "\Phi_E"] & Gr(2, H^0(C,E)^\star) & \end{tikzcd} \tag{3.7}$$

where U is the tautological bundle and $i : U \to H^0(C, E)^\star \otimes \mathcal{O}_G$ is the inclusion of U into the trivial bundle on $Gr(2, H^0(C, E)^\star)$ with fibre $H^0(C, E)^\star$. The map β is defined on the fibres of $E^\star$ by $\beta : E_p^\star \mapsto \Phi_E(p)_{\Phi_E(p)}$. The maps π denote projection down the bundles.

We now find a relationship between α and β. Given any $\sigma \in \Sigma$ we have $\alpha : \sigma \mapsto \mathbb{P}H^0(\Sigma, \mathcal{O}(1)(-\sigma))^\perp$. Consider α restricted to the fibre over $p \in C - \{D_1 + D_2\}$. From our discussion earlier (see results (3.4) and (3.5)) we have:

$$\begin{aligned} \mathrm{im}(\alpha|_p) &= \{\mathbb{P}H^0(\Sigma, \mathcal{O}(1)(-\sigma))^\perp \mid \sigma \in \mathbb{P}E_p^\star\} \\ &\cong \mathbb{P}H^0(C, \pi_\star(\mathcal{O}(1)(-\sigma)))^\perp \\ &\cong \mathbb{P}H^0(C, E(-p))^\perp \\ &\cong \mathbb{P}\beta(E_p^\star). \end{aligned} \tag{3.8}$$

In general, for all $p \in C$ we have $\mathrm{im}(\alpha|_p) \hookrightarrow \mathbb{P}(\beta(E_p^\star))$. At each point $p \in C - \{D_1 + D_2\}$ the map β takes the fibre $E_p^\star$ to a 2-plane in $H^0(C, E)^\star \otimes_G |_{\Phi_E(p)}$; which is represented by the line $\alpha(\Sigma_p) \subset \mathbb{P}H^0(C, E)^\star$. We would like to show that:

$$\alpha(\Sigma_p) \subset q_\phi \Leftrightarrow \mathbb{P}\beta(E_p^\star) \in \mathbb{G}r(1, q_\phi). \tag{3.9}$$

By (3.8) we know that the condition holds for $p \in C - \{D_1 + D_2\}$. Now suppose $p \in D_1 + D_2$ then take a sequence $\{p_i\}_{i=1}^{\infty} \subset C - \{D_1 + D_2\}$ converging to p. If $\alpha(\Sigma_p) \subset q_\phi$ then $\mathbb{P}H^0(C, E(-p_i))^\perp = \mathbb{P}\beta(E^\star_{p_i}) \in \mathbb{G}r(1, q_\phi)$ so

$$\mathbb{P}\beta(E^\star_p) = \mathbb{P}\lim_{i\to\infty} H^0(C, E(-p_i))^\perp \in \mathbb{G}r(1, q_\phi).$$

The converse is clear because $\operatorname{im}(\alpha|_p) \hookrightarrow \mathbb{P}(\beta(E^\star_p))$.

A form on U may be defined by restriction of the quadratic form ϕ on the fibres $H^0(C, E)$. Define Ψ_E to be this restricion map:

$$\Psi_E : \operatorname{Sym}^2 H^0(C, E) \otimes \mathcal{O}_G \to H^0(G, \operatorname{Sym}^2 U^\star) \otimes \mathcal{O}_G,$$

where $\Psi_E : \phi \mapsto i^\star \circ \phi \circ i$.

We can now rephrase injectivity of μ in terms of the Grassmannian by our characteristation using the ruled surface Σ. First we identify $q_\phi \subset \mathbb{P}H^0(C, E)^\star$ with the corresponding quadric in $\mathbb{P}H^0(\Sigma, \mathcal{O}(1))^\star$ by using our isomorphism (3.5). Then we have:

$$\begin{aligned} \alpha(\Sigma) \subset q_\phi &\Leftrightarrow \mathbb{P}\beta(E^\star_p) \subset q_\phi, \text{ for all } p \in C \\ &\Leftrightarrow \Psi_E(\phi)(\Phi_E(p)) = 0, \text{ for all } p \in C \\ &\Leftrightarrow \Phi_E(C) \text{ lies in the zero locus of } \Psi_E(\phi). \end{aligned} \tag{3.10}$$

We now determine the zero locus of $\Psi_E(\phi)$. Given a point $\Lambda \in Gr(2, H^0(C, E)^\star)$, $\Psi_E(\phi)(\Lambda) = 0$ if and only if the fibre of U at Λ embedded in $H^0(C, E)^\star$ lies in the zero locus of ϕ. The fibre of U at Λ is just Λ itself so the zero locus of $\Psi_E(\phi)$ is $\mathbb{G}r(1, q_\phi)$. Using (3.10) we get that:

$$\mu(\phi) = 0 \Leftrightarrow \alpha(\Sigma) \subset q_\phi \Leftrightarrow \Phi_E(C) \subset \mathbb{G}r(1, q_\phi).$$

We conclude that μ is injective if and only if $\Phi_E(C)$ is not contained in $\mathbb{G}r(1, q)$ for any quadric q. $\square$

3.2 Injectivity of the Petri map

The result that we will be working towards in the following three sections, and in fact the main result of the thesis is the following.

Theorem 1.3.7 *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2, K)$ stable where $h^0(C, E) \leq 6$. Then the Petri map:*

$$\mu : Sym^2 H^0(C, E) \to H^0(C, Sym^2 E),$$

is injective.

Our method for proving this theorem is to look at the image of C under Φ_E and show that it cannot lie in $\mathbb{G}r(1, q)$ for any quadric $q \subset \mathbb{P}H^0(C, E)^\star$. We start by proving the result for smooth quadrics in $\mathbb{P}H^0(C, E)^\star$. The following proposition is shown in section 3.3.

Proposition 3.2.1. *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2, K)$ stable and generically globally generated with $3 \leq h^0(C, E) \leq 6$. Then $\Phi_E(C)$ cannot lie in a $\mathbb{G}r(1, q)$ for any smooth quadric $q \in \mathbb{P}H^0(C, E)^\star$.*

Bertram-Feinberg [4] prove this result for $h^0(C, E) \leq 5$ without genus restriction. However, their proof is only valid for globally generated bundles and is adapted in section 3.3 to hold for generically globally generated bundles.

We now deal with the singular quadrics in $\mathbb{P}H^0(C, E)^\star$. For E globally generated and q a singular quadric Bertram-Feinberg [4] prove that $\Phi_E(C) \not\subseteq \mathbb{G}r(1, q)$ for $h^0(C, E) \leq 4$ and remark that the method of proof may be extended to $h^0(C, E) = 5$.

Let q be a quadric such that $\text{rank}(q) < h^0(C, E)$. We denote the singular locus of q by $\ker(q) \cong \mathbb{P}^k$, where $k = h^0(C, E) - \text{rank}(q) - 1$. Using this singular locus we define a subvariety of $Gr(2, H^0(C, E)^\star)$.

Definition 3.2.2. *Denote the vertex variety by*

$$\mathcal{V}(q) := \{l \in \mathbb{G}r(1, q) \mid l \cap \mathbb{P}ker(q) \neq \emptyset\}.$$

We consider two different configurations that $\Phi_E(C)$ can take if $\Phi_E(C) \subset \mathbb{G}r(1,q)$. We have that either $\Phi_E(C)$ intersects $\mathcal{V}_E$ in only a finite number of points or lies entirely within $\mathcal{V}_E$.

In the first case we prove the following proposition in section 3.4.

Proposition 3.2.3. *Let $E \in \mathcal{SU}(2,K)$ be stable and generically globally generated with $4 \leq h^0(C,E) \leq 7$. If the image of C under $\Phi_E : C \to Gr(2, H^0(C,E)^\star)$ lies in $\mathbb{G}r(1,q)$ for a singular quadric q such that $\Phi_E(C) \not\subseteq \mathcal{V}_E$; then C is nongeneric for $g \leq 11$.*

Note that in the case $h^0(C,E) = 3$, if $\Phi_E(C) \subset \mathbb{G}r(1,q)$ for a singular conic $q \subset \mathbb{P}H^0(C,E)^\star$ then necessarily $\Phi_E(C) \subset \mathcal{V}_E$. To prove Proposition 3.2.3 we factor out the singular locus of q and appeal to Proposition 3.3.1; a more general version of Proposition 3.2.1 shown in section 3.2.1.

In section 3.5 we prove (subject to Assumption 4.2.1) a proposition that deals with the outstanding case that $\Phi_E(C) \subseteq \mathcal{V}_E$.

Proposition 3.2.4. *Let $E \in \mathcal{SU}(2,K)$ be stable and generically globally generated with $3 \leq h^0(C,E) \leq 6$. If the image of C under $\Phi_E : C \to Gr(2, H^0(C,E)^\star)$ lies in $\mathbb{G}r(1,q)$ for a singular quadric q such that $\Phi_E(C) \subset \mathcal{V}_E$; then C is nongeneric for genus $g \leq 11$.*

To study this situation we generalise an idea used by Bertram and Feinberg [4] when they examined rank 3 quadrics in $\mathbb{P}^3$.

3.3 The curve $\Phi_E(C)$ lies in a smooth quadric

In this section we prove Proposition 3.2.1. We follow the method of proof introduced by Bertram-Feinberg [4]. Let $\boldsymbol{P}$ be the Plücker embedding of the Grassmannian $Gr(2, H^0(C,E)^\star)$ in $\mathbb{P}(\bigwedge^2 H^0(C,E)^\star)$. Composing with the map Φ_E from the curve to the Grassmannian gives:

$$C \overset{\Phi_E}{\to} Gr(2, H^0(C,E)^*) \overset{P}{\hookrightarrow} \mathbb{P}(\bigwedge^2 H^0(C,E)^*). \tag{3.11}$$

Let U be the tautological bundle on the Grassmannian. The Plücker embedding is given by the line bundle $\det(U^*) = \bigwedge^2 U^*$, so we have:

$$(P \circ \Phi_E)^* \mathcal{O}(1) = \Phi_E^* P^* \mathcal{O}(1) = \Phi_E^* \det(U^*) = \det(\Phi_E^* U^*). \tag{3.12}$$

In [4] the assumption $\Phi_E^* U^* = E$ is used to determine the pullback of the hyperplane bundle to the curve. However, $\Phi_E^* U^* = E$ if and only if E is globally generated. For E generically globally generated a different statement is required.

Using the same notation as (3.1) and (3.2) let D_i be the divisor of points on C for which the evaluation map ϵ has rank $(2-i)$. The pullback of U^* sits in the exact sheaf sequence:

$$0 \to \Phi_E^* U^* \to E \to \mathcal{F} \to 0 \tag{3.13}$$

where $\mathcal{F}$ is a torsion sheaf supported on

$$\{p \in C : \operatorname{im}(\epsilon|_p) \ncong \mathbb{C}^2\} = D_1 \cup D_2.$$

The injective map in sequence (3.13) is given by sending the sheaf of sections of $\Phi_E^* U^*$ to the sheaf of sections of E. From this sequence we deduce the determinant of $\Phi_E^* U^*$:

$$\det(\Phi_E^* U^*) = \det(E)(-D) = K(-D), \tag{3.14}$$

where D is an effective divisor supported on $D_1 \cup D_2$.

From equation (3.12) we therefore have:

$$(P \circ \Phi_E)^* \mathcal{O}(1) = \det(\Phi_E^* U^*) = K(-D). \tag{3.15}$$

Following [4] we assume that $\Phi_E(C) \subset \mathbb{G}r(1,q)$ and describe $\det(U^*)$ restricted to $\mathbb{G}r(1,q)$. The pullback of this bundle imposes conditions on the curve that force it to be nongeneric.

In fact we can prove a stronger result than Proposition 3.2.1 by generalising to a larger class of morphism $\Phi : C \to Gr(2, \mathbb{C}^h)$ such that:

$$\Phi^* H^0(G, U^*) \cong H^0(C, \Phi^* U^*) \cong \mathbb{C}^h, \tag{3.16}$$

and

$$0 \to \Phi^* U^* \to E \to \mathcal{F} \to 0, \tag{3.17}$$

where $\mathcal{F}$ is a torsion sheaf. It is clear that the set of morphisms satisfying these conditions contains Φ_E. We now state Proposition 3.2.1 generalised to the morphism Φ.

Proposition 3.3.1. *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2, K)$ stable and generically globally generated. Suppose $\Phi : C \to Gr(2, \mathbb{C}^h)$ is a morphism satisfying conditions* (3.16) *and* (3.17). *For $3 \leq h \leq 6$ there are no smooth quadrics $q \subset \mathbb{P}^{h-1}$ such that $\Phi(C) \subset \mathbb{G}r(1, q)$.*

This stronger result will be used to prove Propostion 3.2.3. The proofs of Proposition 3.2.1 for $3 \leq h^0(C, E) \leq 5$ given in [4] hold true for Proposition 3.3.1 although some slight modification is required in the cases $h = 4, 5$ to take account of generic global generation.

For the case $h = 3$ we can relax the condition of genericity and show the result for all curves.

Lemma 3.3.2. *Let C be an algebraic curve and $\Phi : C \to Gr(2, \mathbb{C}^3)$ a morphism satisfying conditions 3.16 and 3.17. There is no smooth conic $q \in \mathbb{P}^2$ such that $\Phi(C) \subset \mathbb{G}r(1, q)$.*

Proof. A smooth quadric $q \subset \mathbb{P}^2$ does not contain any lines so $\mathbb{G}r(1, q) = \emptyset$; therefore $\Phi(C) \not\subset \mathbb{G}r(1, q)$. □

The next two lemmas cover the cases $h = 4, 5$.

Lemma 3.3.3. *Let $\Phi : C \to Gr(2, \mathbb{C}^4)$ be a morphism satisfying conditions* (3.16) *and* (3.17). *If $\Phi(C) \subset \mathbb{G}r(1, q)$ for $q \subset \mathbb{P}^3$ a smooth quadric then C is nongeneric.*

Proof. Suppose that $\Phi(C) \subset \mathbb{G}r(1,q)$ for a smooth quadric $q \subset \mathbb{P}^3$. The quadric q is a doubly ruled surface whose lines are given by the disjoint union of two lines, that is to say $\mathbb{G}r(1,q) = \mathbb{P}^1 \sqcup \mathbb{P}^1$. The Plücker image of $\mathbb{G}r(1,q)$ has degree 4, therefore the image of each line in $\mathbb{G}r(1,q)$ has degree 2 and so is a conic. The morphism Φ maps C to one of the disjoint lines in $\mathbb{G}r(1,q)$ so $(\boldsymbol{P} \circ \Phi)(C) = C_0$, a conic in Plücker space. The pullback of the hyperplane bundle on $(\boldsymbol{P} \circ \Phi)(C)$ to $\Phi(C)$ will be $\mathcal{O}_{\mathbb{P}^1}(2)$, which gives:

$$\det(\Phi^\star U^\star) = (\boldsymbol{P} \circ \Phi)^\star \mathcal{O}(1) = \Phi^\star \mathcal{O}_{\mathbb{P}^1}(2) = (\Phi^\star \mathcal{O}_{\mathbb{P}^1}(1))^{\otimes 2}. \tag{3.18}$$

Moreover, $r(\Phi^\star \mathcal{O}_{\mathbb{P}^1}(1)) \geq 1$. Otherwise the image of C under Φ is a point so $\Phi(C) \subset Gr(2, \mathbb{C}^3)$ and we conclude that $\Phi^\star H^0(G, U^\star) \cong \mathbb{C}^3$, which contradicts condition (3.16).

Taking determinants in (3.17) gives $\det(\Phi^\star U^\star) = K(-D)$ for some effective divisor D. Putting $L = \Phi^\star \mathcal{O}_{\mathbb{P}^1}(1)$, we have from (3.18)

$$K = L^{\otimes 2}(D). \tag{3.19}$$

The Petri map of L now becomes

$$\mu : H^0(C, L) \otimes H^0(C, L(D)) \to H^0(K) \tag{3.20}$$

and this map cannot be injective since $r(L) \geq 1$. In fact, if s_1, s_2 are independent sections of L and t is a non-zero section of $\mathcal{O}(D)$, then $\mu(s_1 \otimes s_2 t) = \mu(s_2 \otimes s_1 t)$. This contradicts the genericity of C by Theorem 1.1.4. □

Lemma 3.3.4. *Let $\Phi : C \to Gr(2, \mathbb{C}^5)$ be a morphism satisfying conditions* (3.16) *and* (3.17). *If $\Phi(C) \subset \mathbb{G}r(1,q)$ for $q \subset \mathbb{P}^4$ a smooth quadric then C is nongeneric.*

Proof. For a contradiction suppose that $\Phi(C) \subset \mathbb{G}r(1,q)$ for a smooth quadric $q \subset \mathbb{P}^4$. The lines in q may be identified with $\mathbb{P}^3$ and the restriction of the Plücker map to $\mathbb{G}r(1,q)$ is the Veronese embedding:

$$\boldsymbol{P} : \mathbb{P}^3 \overset{|\mathcal{O}(2)|}{\hookrightarrow} \mathbb{P}(\bigwedge^2 \mathbb{C}^5) \cong \mathbb{P}^9. \tag{3.21}$$

We therefore get the expression:

$$\det(\Phi^* U^*) = (\Phi \circ P)^* \mathcal{O}(1) = \Phi^* \mathcal{O}_{\mathbb{P}^3}(2) = (\Phi^* \mathcal{O}_{\mathbb{P}^3}(1))^{\otimes 2}. \tag{3.22}$$

Again by our definition of Φ we must have $r(\Phi^* \mathcal{O}_{\mathbb{P}^3}(1)) \geq 1$. We now complete the proof exactly as for Lemma 3.3.3 . □

Having explained the proofs of Bertram-Feinberg we apply their methods to the case when $h = 6$. Unlike the previous cases though, we must restrict the genus of the curve.

Proposition 3.3.5. *Let C be a generic curve of genus $g \leq 11$ and $\Phi : C \to Gr(2, \mathbb{C}^6)$ a morphism satisfying conditions* (3.16) *and* (3.17). *There are no smooth quadrics $q \subset \mathbb{P}^5$ such that $\Phi(C) \subset \mathbb{G}r(1, q)$.*

We start by describing $\mathbb{G}r(1, q)$, where $q \subset \mathbb{P}^5$ is a smooth quadric - that is to say the Klein quadric. Recall that the Klein quadric is the image of $Gr(2, \mathbb{C}^4)$ under the Plucker embedding. There are two disjoint families of 2-planes in q, the first parametrised by $\mathbb{P}^3$ and the second by $(\mathbb{P}^3)^*$. These are known respectively as the alpha and beta planes where:

$$\alpha_p := \{\text{lines passing through } p\},$$
$$\beta_\pi := \{\text{lines lying in } \pi\}.$$

We can see that planes in the same family intersect in points or planes, whereas planes in opposite families either intersect in a line or not at all. Hence a line l lying in the quadric is the intersection of an alpha and beta plane; $l = \alpha_p \cap \beta_\pi = \{\text{lines lying in } \pi \text{ that pass through } p\}$, for some $p \in \mathbb{P}^3$ and $\pi \in (\mathbb{P}^3)^*$. The Grassmannian $\mathbb{G}r(1, q)$ can thus be identified with an incidence variety:

$$\begin{aligned} \mathbb{G}r(1, q) &\quad \overset{\sim}{\longleftrightarrow} I = \{(p, \pi) : p \in \pi\} \subseteq \mathbb{P}^3 \times (\mathbb{P}^3)^* \\ l = \alpha_p \cap \beta_\pi &\overset{1:1}{\longleftrightarrow} (p, \pi). \end{aligned} \tag{3.23}$$

We can illustrate the situation with the following diagram:

$$\begin{tikzcd} & \mathbb{P}^3 & \\ C \arrow[r, "\Phi"] & I \arrow[u, "pr_1"] \arrow[d, "pr_2"'] \arrow[r, "P"] & \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \\ & (\mathbb{P}^3)^\star & \end{tikzcd}$$

where pr_1 and pr_2 are projections to the first and second factors of I.

We now identify the Picard group of I.

Lemma 3.3.6. *Let q be a smooth quadric lying in $\mathbb{P}^5$, and I the incidence variety isomorphic to $\mathbb{G}r(1,q) \subset \mathbb{G}r(1,\mathbb{P}^5)$; then:*

$$Pic(I) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. To find the Picard group of I it is required to find information about the cohomology of I since $\mathrm{Pic}(I) = H^1(I, \mathcal{O}^\star)$. The method of proof will be to relate the cohomology of I with that of $\mathbb{P}^3 \times (\mathbb{P}^3)^\star$ by using the Lefschetz hyperplane theorem. Recall that this states that for M an n-dimensional compact manifold and $V \subset M$ a smooth hypersurface then the map

$$H^q(M,\mathbb{Q}) \to H^q(V,\mathbb{Q})$$

induced by the inclusion $i \hookrightarrow M$ is an isomorphism for $q \leq n-2$ and injective for $q = n-1$.

To start with we must show that I is a hypersurface of $\mathbb{P}^3 \times (\mathbb{P}^3)^\star$. Let $\{v_0, \ldots, v_3\}$ be a basis for $\mathbb{C}^4$. Then $\mathbb{P}^3 = \mathbb{P}\langle v_0, \ldots, v_3\rangle$ and $(\mathbb{P}^3)^\star = \mathbb{P}\langle f_0, \ldots, f_3\rangle$ where $f_i(v_j) = \delta_{i,j}$. Let $p = [v] \in \mathbb{P}^3$ and $\pi = \ker(f) \in (\mathbb{P}^3)^\star$; v and f are defined by:

$$v = \sum_{i=0}^{3} \lambda_i v_i \qquad \text{and} \qquad f = \sum_{i=0}^{3} \mu_j f_j, \tag{3.24}$$

for some $\lambda_i, \mu_j \in \mathbb{C}$. Then:

$$\begin{aligned} p \in \pi \Leftrightarrow 0 = f(v) &= \sum_{j=0}^{3} \mu_j f_j(\sum_{i=0}^{3} \lambda_i v_i) \\ &\Leftrightarrow \mu_0\lambda_0 + \cdots + \mu_3\lambda_3 = 0. \end{aligned} \tag{3.25}$$

The incidence variety I is the zero locus of the homogenous polynomial (3.25) and so is a hypersurface of $\mathbb{P}^3 \times (\mathbb{P}^3)^\star$.

By considering the cohomology sequence associated with the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0,$$

on both I and $\mathbb{P}^3 \times (\mathbb{P}^3)^\star$ a commutative diagram can be constructed. To ease congestion on the page let $M := \mathbb{P}^3 \times (\mathbb{P}^3)^\star$. We then have:

$$\begin{tikzcd} \dots \arrow[r] & H^1(M,\mathcal{O}) \arrow[r] \arrow[d] & H^1(M,\mathcal{O}^\star) \arrow[r] \arrow[d] & H^2(M,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(M,\mathcal{O}) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & H^1(I,\mathcal{O}) \arrow[r] & H^1(I,\mathcal{O}^\star) \arrow[r] & H^2(I,\mathbb{Z}) \arrow[r] & H^2(I,\mathcal{O}) \arrow[r] & \dots \end{tikzcd} \tag{3.26}$$

where the vertical maps are restrictions. By the Künneth formula $H^2(\mathbb{P}^3\times(\mathbb{P}^3)^\star, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and $H^2(\mathbb{P}^3 \times (\mathbb{P}^3)^\star, \mathcal{O}) = 0 = H^1(\mathbb{P}^3 \times (\mathbb{P}^3)^\star, \mathcal{O})$. The Lefschetz hyperplane theorem tells us that $H^2(I,\mathbb{Z}) \cong H^2(\mathbb{P}^3 \times (\mathbb{P}^3)^\star, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. The grading of cohomology into holomorphic and antiholomorphic forms will be preserved by the isomorphisms given by restriction. Hence $H^2(I,\mathcal{O}) \cong H^2(\mathbb{P}^3 \times (\mathbb{P}^3)^\star, \mathcal{O}) = 0$ and $H^1(I,\mathcal{O}) \cong H^1(\mathbb{P}^3 \times (\mathbb{P}^3)^\star, \mathcal{O}) = 0$. Substituting this information into diagram (3.26) gives us that $\mathrm{Pic}(I) = H^1(I, \mathcal{O}^\star) = \mathbb{Z} \oplus \mathbb{Z}$. □

Lemma 3.3.7. *Let U be the tautological bundle on $Gr(2,\mathbb{C}^6)$. For any smooth quadric $q \subset \mathbb{P}^5$ with I the incidence variety isomorphic to $\mathbb{G}r(1,q)$:*

$$det(U|_I) = \mathcal{O}(-1,-1).$$

Proof. We refer back to (3.23) and subsequent comments for the description of the variety I. Further let $V \cong \mathbb{C}^4$ be the underlying vector space such that:

$$I = \{(p, \pi) \in \mathbb{P}(V) \times \mathbb{P}(V^\star) : p \in \pi\} \subseteq Gr(2, \bigwedge^2 V). \tag{3.27}$$

From the result that $\mathrm{Pic}(I) \cong \mathbb{Z} \oplus \mathbb{Z}$ (Lemma 3.3.6) we define $\mathcal{O}(a, b) := pr_1^\star \mathcal{O}_{\mathbb{P}(V)}(a) \otimes pr_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(b)$. Let F_p be the fibre over $p \in \mathbb{P}(V)$ and F_π the fibre over $\pi \in \mathbb{P}(V^\star)$. Restricting to the fibres, we get $\mathcal{O}(a, b)|_{F_p} = \mathcal{O}_{F_p}(b)$ and $\mathcal{O}(a, b)|_{F_\pi} = \mathcal{O}_{F_\pi}(a)$.

Now $\det(U^\star)|_I \in \mathrm{Pic}(I)$ so $\det(U^\star)|_I = \mathcal{O}(a, b)$ for some $a, b \in \mathbb{Z}$. Moreover,

$$\det(U^\star)|_I = \mathcal{O}(a, b) \Leftrightarrow \begin{cases} \det(U^\star)|_{F_p} = \mathcal{O}_{F_p}(b) \\ \det(U^\star)|_{F_\pi} = \mathcal{O}_{F_\pi}(a) \end{cases} \tag{3.28}$$

To determine $\det(U^\star)|_I$ we restrict to the fibres F_p and F_π. Recalling that $(p, \pi) \in I$ refers to the line intersection of α_p and β_π, the fibres may be described as

$$F_p = \{\text{lines in } \alpha_p\}$$
$$F_\pi = \{\text{lines in } \beta_\pi\}.$$

We characterise the fibre $F_p \subseteq Gr(2, \bigwedge^2 V)$. Let $x \in V$ be a representative of $p \in \mathbb{P}(V)$, that is $p \in [x]$. Then $\alpha_p \cong \{\text{lines passing through } p\} = \mathbb{P}(V/\langle x \rangle)$.

The embedding $i : V/\langle x \rangle \hookrightarrow \bigwedge^2 V$ given by $i : y \mapsto x \wedge y$ extends naturally to $Gr(2, V/\langle x \rangle) \hookrightarrow Gr(2, \bigwedge^2 V)$. We know that $F_p = \{\text{ lines in } \alpha_p\} \cong Gr(2, V/\langle x \rangle)$, therefore F_p is the image of $Gr(2, V/\langle x \rangle) \hookrightarrow Gr(2, \bigwedge^2 V)$.

Now consider the fibre $F_\pi \subseteq Gr(2, \bigwedge^2 V)$. Given $\pi \in \mathbb{P}(V^\star)$ there exists $\lambda \in V^\star$ such that $\pi = \ker(\lambda)$. Then $\beta_\pi \cong \{\text{lines lying in } \pi\} = Gr(2, \ker(\lambda))$. Now $F_\pi = \{\text{lines in } \beta_\pi\}$, if $\boldsymbol{P}$ is the Plücker embedding then

$$F_\pi \cong Gr(2, \boldsymbol{P}(Gr(2, \ker(\lambda)) = Gr(2, \bigwedge^2 \ker(\lambda)). \tag{3.29}$$

The inclusion $\bigwedge^2(\ker(\lambda)) \subseteq \bigwedge^2 V$ extends to $Gr(2, \bigwedge^2 \ker(\lambda)) \subseteq Gr(2, \bigwedge^2 V)$. Then $F_\pi = Gr(2, \bigwedge^2 \ker(\lambda)) \subseteq Gr(2, \bigwedge^2 V)$.

Now we move to looking at $\det(U^\star)|_I$ by restricting to the fibres. Let U_p be the tautological bundle on $Gr(2, V/\langle x\rangle) \cong F_p$. The bundle U restricts to U_p on the fibre F_p. Noting that $F_p \cong Gr(1, (V/\langle x\rangle)^\star) = \mathbb{P}((V/\langle x\rangle)^\star)$ the tautological sequence on $Gr(2, V/\langle x\rangle)$ is

$$0 \to U_p \to V/\langle x\rangle \otimes \mathcal{O} \to \mathcal{O}_{F_p}(1) \to 0. \tag{3.30}$$

Therefore $\det(U_p^\star)|_{F_p} = \mathcal{O}_{F_p}(1)$.

Let U_π be the tautological bundle on $Gr(2, \bigwedge^2 \ker(\lambda)) = F_\pi$. The bundle U restricts to U_π on the fibre F_π. Knowing that $F_\pi \cong Gr(1, (\bigwedge^2 \ker(\lambda))^\star) = \mathbb{P}(\bigwedge^2 \ker(\lambda)^\star)$, the tautological sequence on $Gr(2, \bigwedge^2 \ker(\lambda))$ is

$$0 \to U_\pi \to \bigwedge^2 \ker(\lambda) \otimes \mathcal{O} \to \mathcal{O}_{F_\pi}(1) \to 0. \tag{3.31}$$

Therefore $\det(U^\star)|_{F_\pi} = \det(U_\pi^\star) = \mathcal{O}_{F_\pi}(1)$. By equation (3.28) we have

$$\det(U^\star)|_I = \mathcal{O}(1,1) \tag{3.32}$$

□

Suppose now that $\Phi(C) \subset \mathbb{G}r(1,q)$, then Lemma 3.3.7 tells us $\det(U^\star|_I) = \mathcal{O}(1,1)$. Defining $\phi_i := pr_i \circ \Phi$ gives:

$$(\boldsymbol{P} \circ \Phi)^\star \mathcal{O}(1) = \Phi^\star \det(U^\star|_I) = \Phi^\star \mathcal{O}(1,1) = \phi_1^\star \mathcal{O}(1) \otimes \phi_2^\star \mathcal{O}(1). \tag{3.33}$$

We have the sequence:

$$0 \to \Phi^\star U^\star \to E \to \mathcal{F} \to 0, \tag{3.34}$$

where $\mathcal{F}$ is a torsion sheaf. As before we take determinants to obtain, for a suitable effective divisor D,

$$K(-D) = \det(\Phi^\star U^\star) = (\boldsymbol{P} \circ \Phi)^\star \mathcal{O}(1) = \phi_1^\star \mathcal{O}(1) \otimes \phi_2^\star \mathcal{O}(1). \tag{3.35}$$

This gives us an upper bound on the degrees of $\phi_1^\star \mathcal{O}(1)$ and $\phi_2^\star \mathcal{O}(1)$:

$$2g - 2 = \deg(K) \geq \deg(\phi_1^\star \mathcal{O}(1)) + \deg(\phi_2^\star \mathcal{O}(1)). \tag{3.36}$$

The idea of the following proof is to show that $r(\phi_1^\star\mathcal{O}(1))$ and $r(\phi_2^\star\mathcal{O}(1))$ are sufficiently large to ensure that either $\rho(\phi_1^\star\mathcal{O}(1)$ or $\rho(\phi_2^\star\mathcal{O}(1))$ is negative. In Lemma 3.3.8 it is shown that neither of the ϕ_i maps C to a line. Subsequently in Lemma 3.3.9 we prove that one of the $\phi_i(C)$ does not lie in a hyperplane, and the proof of Proposition 3.3.5 is concluded by Lemma 3.3.10.

Lemma 3.3.8. *Let C be a generic curve and $\Phi : C \to Gr(2,\mathbb{C}^6)$ a morphism satisfying conditions* (3.16) *and* (3.17)*. Let $q \subset \mathbb{P}^5$ be a smooth quadric and $\Phi(C) \subset \mathbb{G}r(1,q) \cong I$; then the $\phi_i(C)$ cannot lie on a line.*

Proof. Assume for a contradiction that $\phi_1 : C \to l \cong \mathbb{P}^1$ (the proof is equivalent whichever of the ϕ_i maps C to a line). There are two possibilities, either $\phi_1(C)$ is a point or a line.

In the first case suppose $\phi_1(C) = p$. Then all the projective lines corresponding to points of $\Phi(C)$ lie in $\alpha_p \subset q$. That is to say $\Phi(C) \subset Gr(2,\mathbb{C}^3)$; therefore $H^0(C,\Phi^\star U^\star) \subset \mathbb{C}^3$ which contradicts condition (3.16).

Now consider the possibility $l \cong \mathbb{P}^1$. Choose a point $(p,\pi) \in \Phi(C)$ which corresponds to those lines in $\mathbb{P}^3$ that pass through p and lie in π. However, p must lie on the line l, so in particular all the lines given by points of $\Phi(C)$ meet l. The lines meeting l constitute a Schubert cycle in $\mathbb{G}r(1,\mathbb{P}^3)$:

$$\{\text{lines meeting } l\} \cong \sigma_1(l).$$

However, $\sigma_1(l) = T_l Gr(2,4) \cap Gr(2,4)$ (see Griffiths-Harris [11] page 757); so $\sigma_1(l)$ is the cone over a smooth quadric in $\mathbb{P}^3$, with vertex l. Therefore $\Phi(C) \subset \mathbb{G}r(1,\sigma_1(l))$, so in particular $\Phi(C) \subset Gr(2,\mathbb{C}^5)$. Consequently $H^0(C,\Phi^\star U^\star) \subset \mathbb{C}^5$ which again contradicts condition (3.16). $\square$

Lemma 3.3.9. *Let C be a generic curve and $\Phi : C \to Gr(2,\mathbb{C}^6)$ a morphism satisfying conditions* (3.16) *and* (3.17)*. Let $q \subset \mathbb{P}^5$ be a smooth quadric and $\Phi(C) \subset \mathbb{G}r(1,q) \cong I$; then one of the $\phi_i(C)$ does not lie in a hyperplane.*

Proof. For a contradiction assume that $\phi_1(C) \subset H$ and $\phi_2(C) \subset h$ where H and h are hyperplanes in $\mathbb{P}^3$ and $(\mathbb{P}^3)^\star$ respectively. In the proof we will be interchangeably referring to objects in their ambient space or duals depending on the context. The constraint that both $\phi_1(C)$ and $\phi_2(C)$ lie on hyperplanes gives us information about the degrees of $\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1)$ and $\phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1)$. We use the definition that

$$\begin{aligned} d_1 &:= \deg(\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1)) \\ d_2 &:= \deg(\phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1)). \end{aligned}$$

Consider $\phi_1 : C \to H \cong \mathbb{P}^2$, then the Brill-Noether number of $\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1)$ is

$$\begin{aligned} 0 \leq \rho(2, d_1) &= g - (2+1)(g - d_1 + 2) = 3d_1 - 2g - 6 \\ \Rightarrow d_1 &\geq \frac{1}{3}(2g + 6). \end{aligned}$$

Hence for generic curves of genus $8 \leq g \leq 11$ we have $d_1 \geq g-1$. Similarly $d_2 \geq g-1$ for generic curves in this genus range. However,

$$\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1) = K(-D) = \det(\Phi^\star U^\star), \tag{3.37}$$

which gives the following inequalities linking the degrees d_1 and d_2

$$2g - 2 = \deg(K) \geq d_1 + d_2 \geq (g-1) + (g-1) \Rightarrow d_1 = d_2 = g - 1. \tag{3.38}$$

Consequently, $\deg(\Phi_E^\star U^\star) = \deg(K(-D^1 - 2D^2)) = 2g - 2$, which implies $D^1 = D^2 = 0$. Therefore $\det(\Phi_E^\star U^\star) = K$ and E is globally generated.

We now look at the possible configurations of the planes α_h and β_H in the quadric. We saw previously that the intersection of α_h and β_H is either empty or a line. The aim of the proof is to find a theta characteristic with sufficient sections to imply that C is non-generic.

Consider the possibility that $\alpha_h \cap \beta_H$ is a line. This condition translates in $\mathbb{P}(V)$ to there being a 1 parameter family of lines passing through h and lying in H. We therefore have $h \in H \subseteq \mathbb{P}(V)$, and in particular $\alpha_h \cap \beta_H = (h, H) \in I$.

We now pick a basis for $\mathbb{P}(V)$ so that we can study the curves $\phi_1(C) \subseteq H$ and $\phi_2(C) \subseteq h$. Let $V = \langle v_0, v_1, v_2, v_3 \rangle$ where $H = \mathbb{P}(\langle v_0, v_1, v_2 \rangle)$ and $h = [v_0]$. Note that in $V^\star = \langle v_0^\star, v_1^\star, v_2^\star, v_3^\star \rangle$, $H = [v_3^\star]$ and $h = \langle v_1^\star, v_2^\star, v_3^\star \rangle$.

Given a point $(p, \pi) \in \Phi(C)$ we consider the projection to $\mathbb{P}(V)$. We know that α_h and α_p correspond to lines through h and p; the two alpha planes will intersect in a point which corresponds to the line joining p and h which we denote by $\overline{ph} \subseteq \mathbb{P}(V)$. Similarly β_H and β_π correspond to the lines lying in hyperplanes H and π, the intersection of β_H and β_π is a point corresponding to a line $H \cap \pi \subseteq \mathbb{P}(V)$.

During the above discussion we assumed that $p \neq h$ and $\pi \neq H$, enabling us to consider lines $\overline{ph}$ and $\overline{\pi H}$. We know that for only a finite number of points will $p = h$ and/or $\pi = H$ because the image of C may not be mapped to point by either ϕ_1 or ϕ_2 (see Lemma 3.3.8). We can therefore assign a line to p (as a proxy for $\overline{ph}$) by taking the limit of $\{\overline{p_i h}\}_{i=1}^{\infty}$ when $\{p_i\}_{i=1}^{\infty} \subseteq \phi_1(C)$ such that $p_i \to p$ as $i \to \infty$. Similarly a line may be assigned to π if $\pi = H$. In the remainder of the proof we will assume lines $\overline{ph}$ and $\overline{\pi H}$ exist, as an appropriate limiting line may always be constructed.

Returning to $\overline{ph}$ and $\pi \cap H$ we can see that both these lines in $\mathbb{P}(V)$ are the same. First of all we know that $p \in \pi$ and $p \in H$ by assumption. On the other hand $h \in \pi$ because $\pi \in h \subseteq \mathbb{P}(V^\star)$, and $h \in H$ by the assmption that α_h and β_H meet in a line. Therefore $p, h \in \pi \cap H$, which implies that $\overline{ph} \subseteq \pi \cap H$ and we conclude that $\overline{ph} = \pi \cap H$. Translating this result to the quadric in $\mathbb{P}(\bigwedge^2 V)$ we have $\alpha_h \cap \alpha_p = \beta_H \cap \beta_\pi \subseteq \alpha_h \cap \beta_H$, that is to say that all lines in the ruled surface defined by $\Phi(C)$ meet the line $\alpha_h \cap \beta_H$.

In order to find a relationship between $\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1)$ and $\phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1)$ we look at the lines in H passing through h (in $\mathbb{P}(V)$) and relate these to lines in h passing through H in $\mathbb{P}(V^\star)$. It will be sufficient to find a relationship between the line $\overline{ph} \subseteq \mathbb{P}(V)$ and $\overline{\pi H} \subseteq \mathbb{P}(V^\star)$. We know that $p = [x]$ for some $x \in \langle v_0, v_1, v_2 \rangle$. We have that $h = [v_0]$ and since we are looking at the two plane $\langle v_0, x \rangle$ we can assume that $x = \lambda_1 v_1 + \lambda_2 v_2$. Previously we noted the equality $\overline{ph} = \pi \cap H \subseteq \mathbb{P}(V)$, this dualises to $\overline{\pi H} = p \cap h \subseteq \mathbb{P}(V^\star)$. This identification will enable us to use our vector description of $\overline{ph}$. The dual 2-planes to p and h in $\mathbb{P}(V^\star)$ are given by

$$\begin{aligned} h = [v_0] &\longrightarrow \mathbb{P}(\langle v_1^\star, v_2^\star, v_3^\star\rangle) \\ p = [x] &\longrightarrow \mathbb{P}(\langle v_0^\star, \lambda_1 v_2^\star - \lambda_2 v_1^\star, v_3^\star\rangle). \end{aligned}$$

Therefore $\overline{\pi H} = p \cap h = \mathbb{P}(\langle \lambda_1 v_2^\star - \lambda_2 v_1^\star, v_3^\star\rangle) = \mathbb{P}(\langle \mathrm{ann}(x), v_3^\star\rangle)$, where we define $\mathrm{ann}(x) := \{f \in \langle v_1^\star, v_2^\star\rangle : f(x) = 0\}$. We therefore have a $1:1$ correspondence

$$\begin{aligned} \{\overline{ph} : p \in \phi_1(C)\} &\longleftrightarrow \{\overline{\pi H} : \pi \in \phi_2(C)\} \\ \overline{ph} = \mathbb{P}(\langle v_0, x\rangle) &\longleftrightarrow \mathbb{P}(\langle \mathrm{ann}(x), v_3^\star\rangle) = \overline{\pi H}. \end{aligned} \tag{3.39}$$

We now look at how we can relate the pullbacks $\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1)$ and $\phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1)$ by using the correspondence above. We do this by parameterising the lines described in (3.39). Let $p_h : H \twoheadrightarrow \mathbb{P}^1$ be the projection of H away from h. Then

$$p_h : \mathbb{P}(\langle v_0, v_1, v_2\rangle) \twoheadrightarrow \mathbb{P}(\langle v_1, v_2\rangle).$$

Pulling back $\mathcal{O}_{\mathbb{P}^1}(1)$ to H using p_h gives $\mathcal{O}_H(1)$. Therefore

$$\phi_1^\star \mathcal{O}_{\mathbb{P}(V)}(1) = \phi_1^\star \mathcal{O}_H(1) = (p_h \circ \phi_1)^\star \mathcal{O}_{\mathbb{P}^1}(1). \tag{3.40}$$

Similarly we take the projection $p_H : h \twoheadrightarrow (\mathbb{P}^1)^\star$ of h away from H. Then

$$p_H : \mathbb{P}(\langle v_1^\star, v_2^\star, v_3^\star\rangle) \twoheadrightarrow \mathbb{P}(\langle v_1^\star, v_2^\star\rangle) \text{ and } \phi_2^\star \mathcal{O}_{\mathbb{P}(V^\star)}(1) = (p_H \circ \phi_2)^\star \mathcal{O}_{(\mathbb{P}^1)^\star}(1).$$

Now consider the map $g : \mathbb{P}(\langle v_1, v_2\rangle) \xrightarrow{\sim} \mathbb{P}(\langle v_1^\star, v_2^\star\rangle)$, given by $g : x \mapsto \mathrm{ann}(x)$. The map g identifies {lines through h in H} $\subseteq \mathbb{P}(V)$ and {lines through H in h} $\subseteq \mathbb{P}(V^\star)$. By the correspondence (3.39) we have the following equality of maps

$$(g \circ p_h \circ pr_1 \circ \Phi) = (p_H \circ pr_2 \circ \Phi). \tag{3.41}$$

Look at the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(\langle v_1^\star, v_2^\star\rangle)$ to $\mathbb{P}(\langle v_1, v_2\rangle)$ by g. Clearly $g^\star\mathcal{O}_{(\mathbb{P}^1)^\star}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore (3.41) gives

$$\begin{aligned}\phi_1^\star\mathcal{O}_{\mathbb{P}(V)}(1) &= (p_h \circ pr_1 \circ \Phi)^\star\mathcal{O}_{\mathbb{P}^1}(1) = (p_h \circ pr_1 \circ \Phi)^\star g^\star\mathcal{O}_{(\mathbb{P}^1)^\star}(1)\\ &= (g \circ p_h \circ pr_1 \circ \Phi)^\star\mathcal{O}_{(\mathbb{P}^1)^\star}(1) = (p_H \circ pr_2 \circ \Phi)^\star\mathcal{O}_{(\mathbb{P}^1)^\star}(1)\\ &= \phi_2^\star\mathcal{O}_{\mathbb{P}(V^\star)}(1).\end{aligned}$$

Hence

$$K = \phi_1^\star\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \phi_2^\star\mathcal{O}_{\mathbb{P}(V^\star)}(1) = (\phi_1^\star\mathcal{O}_{\mathbb{P}(V)}(1))^{\otimes 2},$$

where $r(\phi_1^\star\mathcal{O}_{\mathbb{P}(V)}(1)) \geq 2$, which concludes this part of the proof.

We now look at the possibility that the intersection of α_h and β_H is empty. Translating this condition to $\mathbb{P}(V)$ we have that $h \notin H$. For any $(p, \pi) \in \Phi(C)$, we have $p \in \pi \cap H$ and $h \in \pi$. So $\overline{ph}$ and $\pi \cap H$ are distinct lines in $\mathbb{P}(V)$ defining distinct points of the projectivisation of the fibre $U_{(p,\pi)}$. In this way we obtain two disjoint sections of the projective bundle associated to $U|_{\Phi(C)}$ and hence a decomposition $U|_{\Phi(C)} = L_1 \oplus L_2$. So $\Phi^\star U^\star$ is decomposable. Since E is globally generated, we have $E = \Phi^\star U^\star$, so E is decomposable, which is a contradiction.

□

The following lemma completes the proof of Proposition 3.3.5.

Lemma 3.3.10. *Let C be a generic curve and $\Phi : C \to Gr(2, \mathbb{C}^6)$ be a morphism satisfying conditions* (3.16) *and* (3.17). *Suppose $\Phi(C) \subset \mathbb{G}r(1, q)$ for some smooth quadric $q \subset \mathbb{P}^5$; then $g \geq 12$.*

Proof. By Lemma 3.3.9 we know that one of the ϕ_i does not map into a hyperplane, and from Lemma 3.3.8 that neither of the ϕ_i map onto a line. Suppose that $\phi_1 : C \to \mathbb{P}^2$, then the Brill-Noether number of the linear series giving ϕ_1 is:

$$\rho(2, d) = g - (3)(g - d + 2) = 3d - 2g - 6. \tag{3.42}$$

Given that C is generic tells us that $\rho(2,d) \geq 0$, so we must have

$$d \geq \frac{1}{3}(2g+6). \tag{3.43}$$

We know from equation (3.36) that the linear series giving ϕ_2 has degree at most $2g-2-d$. Moreover it is 3 dimensional so its Brill-Noether number is:

$$\rho(3, 2g-2-d) = g - (4)(g - (2g-2-d)+3) = 5g - 4d - 20. \tag{3.44}$$

Again $\rho(3, 2g-2-d) \geq 0$, which gives us

$$d \leq \frac{1}{4}(5g-20). \tag{3.45}$$

Combining equations (3.43) and (3.45) gives the following inequality:

$$\frac{1}{3}(2g+6) \leq d \leq \frac{1}{4}(5g-20) \Rightarrow g \geq \frac{84}{7} = 12. \tag{3.46}$$

Therefore we must have $g \geq 12$.

Now suppose both ϕ_1 and ϕ_2 are non-degenerate. In this case the Brill-Noether number of the linear series giving ϕ_1 satisfies

$$\begin{aligned} 0 \leq \rho(3,d) = g - (4)(g-d+3) &= 4d - 3g - 12 \\ \Rightarrow d \geq \frac{1}{4}(3g+12). \end{aligned}$$

We again have equation (3.45) holding for ϕ_2. Combining with our inequality for ϕ_1 we get

$$\begin{aligned} \frac{1}{4}(3g+12) \leq d \leq \frac{1}{4}(5g-20) \\ \Rightarrow g \geq \frac{32}{2} = 16. \end{aligned}$$

Hence we must have $g \geq 16$, which concludes our proof.

□

3.4 The curve $\Phi_E(C)$ does not lie in the vertex variety

Proposition 3.2.3 *Let $E \in \mathcal{SU}(2,K)$ be stable and generically globally generated with $4 \leq h^0(E) \leq 7$. If the image of C under $\Phi_E : C \to Gr(2, H^0(C,E)^\star)$ lies in $\mathbb{G}r(1,q)$ for a singular quadric q such that $\Phi_E(C) \not\subseteq \mathcal{V}_E$; then C is nongeneric for $g \leq 11$.*

Proof. Let $k+1$ be the dimension of $V \subset H^0(C,E)^\star$ where $\ker(q) = \mathbb{P}(V)$. The condition that $\Phi_E(C)$ is not contained in the vertex variety means that the 2 plane corresponding to $\Phi_E(x)$ meets V trivially, for $x \in C$ generic. Let π be the projection from V in $H^0(C,E)^\star$. Define D^1 to be those points p of C such that the map $(\pi \circ \epsilon^\star|_p) : E_p^\star \to H^0(C,E)^\star/V$ has rank 1. Similarly let D^2 be the divisor of points $p \in C$ for which $(\pi \circ \epsilon|_p)$ is zero. Recalling the definition of D_1 and D_2 as the divisors where ϵ drops rank (see equations (3.1) and (3.2)), note that $D_1 + D_2 \subset D^1 + D^2$. By taking the composition of Φ_E and π we get a morphism:

$$\Phi : C \xrightarrow{\Phi_E} Gr(2, H^0(C,E)^\star) \xrightarrow{\pi} Gr(2, H^0(C,E)^\star/V). \tag{3.47}$$

In a similar manner to our description of Φ_E for generically globally generated E the map Φ at $p \in D^1 + D^2$ is the limit of $\Phi(p_i)$ as $i \to \infty$ for a sequence $\{p_i\}_{i=0}^\infty \subset C - \{D^1 + D^2\}$ converging to p.

The morphism Φ will map C into $\mathbb{G}r(1,q_0)$ where $q_0 := \pi(q)$ a smooth quadric in $\mathbb{P}(H^0(C,E)^\star/V) \cong \mathbb{P}^{h-1}$; where $h-1 = h^0(C,E) - 2 - k$. Let U_0 be the tautological bundle on $Gr(2, H^0(C,E)^\star/V) = Gr(2, \mathbb{C}^h)$.

We now verify that Φ meets conditions (3.16) and (3.17). Firstly $H^0(C, \Phi_E^\star U^\star) \cong H^0(C,E)^\star$; therefore $H^0(C, \Phi^\star U_0^\star) \cong H^0(C,E)^\star/V$, so (3.16) is satisfied. The pull-back of $U_0^\star$ sits in the following short exact sheaf sequence:

$$0 \to \Phi^\star U_0^\star \to E \to \mathcal{F}_0 \to 0, \tag{3.48}$$

where $\mathcal{F}_0$ is a torsion sheaf. Hence Φ meets condition (3.17).

We have a morphism $\Phi : C \to Gr(2, H^0(C,E)^\star/V) = Gr(2, \mathbb{C}^h)$ that satisfies our conditions and $\Phi(C) \subset \mathbb{G}r(1, q_0)$ for $q_0 \subset \mathbb{P}^{h-1}$ a smooth quadric. By Proposition 3.3.1, if $h \leq 6$ and genus $g \leq 11$ then C is nongeneric. However, $h := h^0(C,E)-1-k$. Therefore if $h^0(C,E) \leq 7$ and $g \leq 11$ then C is nongeneric. □

Looking at the proof of Proposition 3.2.3 we notice that we have actually proved a stronger statement.

Proposition 3.4.1. *Let C be an algebraic curve and $E \in \mathcal{SU}(2,K)$ stable. Suppose that $\Phi_E : C \to \mathbb{G}r(2, H^0(C,E)^\star)$ takes C to $\mathbb{G}r(1,q)$ a singular quadric $q \subset \mathbb{P}H^0(C,E)^\star$, where $dim(V) = k+1$ such that $\Phi_E(C)$ is not contained in $\mathcal{V}(q)$. If Proposition 3.3.1 holds for $h-1 = h^0(C,E) - k - 2$ then C is nongeneric in moduli.*

3.5 The curve $\Phi_E(C)$ lies in the vertex variety

In this section we prove the following result subject to Assumption 4.2.1.

Proposition 3.2.4 *Let $E \in \mathcal{SU}(2,K)$ be stable and generically globally generated with $3 \leq h^0(E) \leq 6$. If the image of C under $\Phi_E : C \to Gr(2, H^0(C,E)^\star)$ lies in $\mathbb{G}r(1,q)$ for a singular quadric q such that $\Phi_E(C) \subset \mathcal{V}_E$; then C is nongeneric for genus $g \leq 11$.*

It is assumed that $\Phi_E(C) \subset \mathcal{V}_E \subset \mathbb{G}r(1,q)$, which is the same as saying that all the 2-planes represented by points of $\Phi_E(C)$ intersect $V \subset H^0(C,E)^\star$ nontrivially where $\mathbb{P}(V) = \ker(q)$. Let $k+1 = \dim(V) = \dim(\ker(q))+1$. Our first step is to impose an upper bound on k that depends on $h^0(C,E)$.

Observe that if $k = h^0(C,E) - 2$ the quadric q is a double hyperplane and for $k = h^0(C,E) - 3$ the quadric is a pair of hyperplanes. In either case the ruled surface must lie in a hyperplane H which is the zero locus of a (nonzero) section $s \in H^0(G, U^\star)$. Therefore $\Phi_E^\star(s)(x) = 0$ for all $x \in C$ so $\Phi_E^\star(s) = 0$. However, $H^0(G,U^\star) \cong \Phi_E^\star H^0(G,U^\star)$ which provides a contradiction. Therefore the bound on

k is:

$$0 \leq k \leq h^0(C,E) - 4. \tag{3.49}$$

In particular this condition means that Proposition (3.2.4) is proved for $h^0(C,E) = 3$.

We deal with $\Phi_E(C) \subset \mathcal{V}_E$ for increasing values of k. For $k = 0$ we follow Bertram-Feinberg [4]. Consider the tautological sequence on the Grassmannian $Gr(2, H^0(C,E)^\star)$:

$$0 \to U \to H^0(C,E)^\star \otimes \mathcal{O} \to Q \to 0, \tag{3.50}$$

where Q is the quotient bundle. Pick $\pi \in H^0(C,E)$ such that $\langle \pi \rangle = V$. Sequence (3.50) gives a map of sections $\Gamma : H^0(C,E)^\star \to H^0(Q)$. Let $s_\pi = \Gamma(\pi) \in H^0(Q)$ and consider its zero locus. Pick $\Lambda \in Gr(2, H^0(C,E)^\star)^\star$ then sequence (3.50) tells us that $s_\pi(\Lambda) = 0$ if and only if $\pi(\Lambda)$ lies in the image of U in $H^0(C,E)^\star \otimes \mathcal{O}$ at Λ. However, $\pi(\Lambda) = \pi$ and $U_\Lambda = \Lambda$, so $\pi \in \Lambda$. We assume that $\Phi_E(C) \subset \mathcal{V}_E$ so every 2-plane represented by a point of $\Phi_E(C)$ contains π, therefore we must have $\Phi_E^\star(s_\pi) = 0$.

The pullback of this sequence (3.50) to the curve is

$$0 \to \Phi_E^\star U \to H^0(C,E)^\star \otimes \mathcal{O} \to Q_E \to 0, \tag{3.51}$$

where Q_E is defined to be $\Phi_E^\star Q$. This induces the following maps on sections:

$$0 \to H^0(C, \Phi_E^\star U) \to H^0(C,E)^\star \to H^0(C, Q_E) \to \dots \tag{3.52}$$

From the fact $\Phi_E^\star(s_\pi) = 0$ we must have that $\pi \in H^0(C,E)^\star$ is the image of a section of $\Phi_E^\star U$. In particular $H^0(C, \Phi_E^\star U) \neq 0$.

Since $h^0(C, \Phi_E^\star U^\star) = h^0(C,E) \geq 3$, $\Phi_E^\star U^\star$ possesses a section with a zero and hence a line subbundle L with $\deg(L) > 0$. From (3.14) we have $\det(\Phi_E^\star U^\star) = K(-D)$ for some effective divisor D. Hence

$$0 \to L \to \Phi_E^\star U^\star \to KL^{-1}(-D) \to 0, \tag{3.53}$$

so dualising gives:

$$0 \to K^{-1}L(D) \to \Phi_E^\star U \to L^{-1} \to 0. \tag{3.54}$$

We then have the bound $h^0(C,\Phi_E^\star U) \leq h^0(C,K^{-1}L(D)) + h^0(C,L^{-1})$. However, $h^0(C,L^{-1}) = 0$ and $h^0(C,\Phi_E^\star U) \neq 0$ so we must have $\deg(K^{-1}L(D)) \geq 0$. Hence $\deg(KL^{-1}(-D)) \leq 0$ which means $h^0(C,KL^{-1}(-D)) \leq 1$. Substituting this value into the upper bound $h^0(C,\Phi_E^\star U^\star) \leq h^0(C,L) + h^0(C,KL^{-1}(-D))$ obtained from sequence (3.53) tells us that $h^0(C,L) \geq h^0(C,\Phi_E^\star U^\star) - 1$. However we know that $h^0(C,\Phi_E^\star U^\star) = h^0(C,E) \geq 4$, therefore $h^0(C,L) \geq 3$. Let d be the degree of L, noting that C is a generic curve we have:

$$0 \leq \rho(L) \leq \rho(2,d) = g - (3)(g-d+2) \Rightarrow d \geq \frac{1}{3}(2g+6). \tag{3.55}$$

Now consider the homomorphism $\Phi_E^\star U^\star \to E$ restricted to L; let $L \to M \subset E$ be the line bundle generated by the image of L. Then we have $\deg(L) \leq \deg(M) \leq g-2$. Hence

$$\frac{1}{3}(2g+6) \leq d \leq g-2 \Rightarrow g \geq 12.$$

This concludes our proof that $\Phi_E(C)$ cannot lie in $\mathcal{V}_E$ for $k=0$.

For higher values of k we generalise the approach used above. Recall the tautological sequence (3.50) and the map $\Gamma : H^0(C,E)^\star \to H^0(G,Q)$. If $\bigwedge^{k+1}\Gamma$ is composed with the multiplication map $\bigwedge^{k+1} H^0(G,Q) \to H^0(G,\bigwedge^{k+1} Q)$; then we obtain:

$$\bigwedge^{k+1} H^0(C,E)^\star \xrightarrow{\bigwedge^{k+1}\Gamma} \bigwedge^{k+1} H^0(G,Q) \xrightarrow{\text{mult}} H^0(G,\bigwedge^{k+1} Q)$$
$$\pi \mapsto s_\pi.$$

If the point $\pi \in \bigwedge^{k+1} H^0(C,E)^\star$ is decomposable then it may be viewed as the image of a $k+1$-plane under the Plücker embedding

$$\boldsymbol{P} : Gr(k+1, H^0(C,E)^\star) \hookrightarrow \bigwedge^{k+1} H^0(C,E)^\star.$$

The decomposable vectors are therefore the ones we are interested in as we are trying to find a description of 2-planes meeting the $k+1$-plane V non-trivially.

Lemma 3.5.1. *Let $\pi \in \bigwedge^{k+1} H^0(C,E)$ be decomposable, then the zero set of s_π represents those planes in $H^0(C,E)^\star$ meeting the $k+1$-plane $V \subset H^0(C,E)^\star$ non-trivially.*

Proof. Let π be the decomposable vector $\sigma_1 \wedge \cdots \wedge \sigma_{k+1} \in \bigwedge^{k+1} H^0(C,E)^\star$, where $\sigma_i \in H^0(C,E)^\star$. Then such a vector represents the $k+1$-plane $\langle \sigma_1, \ldots, \sigma_{k+1} \rangle$ in $H^0(C,E)^\star$. If we define $s_i := \Gamma(\sigma_i) \in H^0(G,Q)$ then

$$s_\pi(\Lambda) = (\text{mult} \circ \bigwedge^{k+1} \Gamma)(\sigma_1 \wedge \cdots \wedge \sigma_{k+1})(\Lambda) = s_1(\Lambda) \wedge \cdots \wedge s_{k+1}(\Lambda).$$

Therefore $s_\pi(\Lambda) = 0$ if and and only if $\{s_1(\Lambda), \ldots, s_{k+1}(\Lambda)\}$ is a linearly dependent set. Consequently we can find $\lambda_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{k+1} \lambda_i s_i(\Lambda) = 0$. By the exactness of sequence (3.50) the preimage of $\sum_{i=1}^{k+1} \lambda_i s_i(\Lambda)$ in $H^0(C,E)^\star \otimes \mathcal{O}$ must come from a point of U_Λ; thus $\sum_{i=1}^{k+1} \lambda_i \sigma_i(\Lambda) \in \Lambda$. However, for a section $\sigma \in H^0(G, H^0(C,E)^\star \otimes \mathcal{O})$ we have $\sigma(\Lambda) = \sigma$. Thus $\sum_{i=1}^{k+1} s_i = \sum_{i=1}^{k+1} \sigma_i(\Lambda) \in \Lambda$, and consequently $\Lambda \cap \langle \sigma_1, \ldots, \sigma_{k+1} \rangle \neq \emptyset$. □

The pull back of s_π lies in $H^0(C, \bigwedge^{k+1} Q_E)$. Following the proof for $k=0$ we construct an exact sequence that $H^0(C, \bigwedge^{k+1} Q_E)$ sits in. We pull back the tautological sequence on $Gr(2, H^0(C,E)^\star)$ to get $0 \to \Phi_E^\star U \to H^0(C,E) \otimes \mathcal{O} \xrightarrow{\gamma} Q_E \to 0$. By taking the $k+1$ th exterior power of γ we obtain the surjective map

$$\bigwedge^{k+1} \gamma : \bigwedge^{k+1} H^0(C,E)^\star \otimes \mathcal{O} \to \bigwedge^{k+1} Q_E.$$

This gives a short exact sequence:

$$0 \to B_{k+1} \to \bigwedge^{k+1} H^0(C,E)^\star \otimes \mathcal{O} \to \bigwedge^{k+1} Q_E \to 0, \tag{3.56}$$

where B_{k+1} is defined by the sequence.

We now return to $\Phi_E^\star(s_\pi)$. It is assumed that $\Phi_E(C) \subset \mathcal{V}_E$, so every 2-plane represented by a point of $\Phi_E(C)$ meets V nontrivially. By Lemma 3.5.1 $\Phi_E(C)$ lies in the zero set of s_π, therefore $\Phi_E^\star(s_\pi) = 0$.

The exactness of the cohomology sequence:

$$0 \to H^0(C, B_{k+1}) \to \bigwedge^{k+1} H^0(C,E)^\star \to H^0(C, \bigwedge^{k+1} Q_E) \to \ldots$$

associated to (3.56) tells us that $H^0(C, B_{k+1})$ is nonzero. In order to use this fact we need to determine the kernel bundle B_{k+1}. To do this a standard result is now stated.

Lemma 3.5.2. *Let F, W and W' be vector bundles over a topological space such that rank$(F) = 2$ and:*

$$0 \to F \xrightarrow{\beta} W \xrightarrow{\gamma} W' \to 0.$$

Taking exterior powers gives:

$$0 \to B_p \to \bigwedge^p W \xrightarrow{\Lambda^p \gamma} \bigwedge^p W' \to 0$$

where:

$$0 \to \bigwedge^2 F \otimes \bigwedge^{p-2} W' \xrightarrow{\phi_p} B_p \xrightarrow{\psi_p} F \otimes \bigwedge^{p-1} W' \to 0.$$

Recalling that $\bigwedge^2 \Phi_E^\star U^\star = \det(\Phi_E^\star U^\star) = K(-D)$, the kernel bundle B_{k+1} defined in equation (3.56) sits in the following exact sequence:

$$0 \to K^{-1}(D) \otimes \bigwedge^{k-1} Q_E \to B_{k+1} \to \Phi_E^\star U \otimes \bigwedge^k Q_E \to 0. \tag{3.57}$$

The cohomology sequence immediately gives an upper bound on the number of sections,

$$h^0(C, B_{k+1}) \leq h^0(C, K^{-1}(D) \otimes \bigwedge^{k-1} Q_E) + h^0(C, \Phi_E^\star U \otimes \bigwedge^k Q_E). \tag{3.58}$$

In particular if $H^0(C, K^{-1}(D) \otimes \bigwedge^{k-1} Q_E) = H^0(C, \Phi_E^\star U \otimes \bigwedge^k Q_E) = 0$ then our result is proved. To show that these cohomology groups vanish we use a result of Narasimhan-Ramanan [20]; Lemma 2.1.

Lemma 3.5.3. *Let V and W be vector bundles over a curve C where $V \not\cong W$. Then:*

1. *if V and W are semistable with $\mu(V) > \mu(W)$ then $H^0(C, V^\star \otimes W) = 0$;*
2. *if V and W are stable with $\mu(V) \geq \mu(W)$ then $H^0(C, V^\star \otimes W) = 0$.*

To use this lemma it is required that $\Phi_E^\star U^\star$ and the exterior powers of Q_E satisfy stability and slope conditions. In Lemma 3.5.5 we prove that $\Phi_E^\star U^\star$ is stable and in Proposition 3.5.7 that the exterior powers are semistable. In order to use part

1 of Lemma 3.5.3 we need $\mu(K(-D)) > \mu(\bigwedge^{k-1} Q_E)$ and $\mu(\Phi_E^\star U^\star) > \mu(\bigwedge^k Q_E)$. We calculate the slopes of $\bigwedge^i Q_E$ in Lemma 3.5.8 and Corollary 3.5.9. When $k = 2$ it transpires that $\mu(\Phi_E^\star U^\star) = \mu(\bigwedge^2 Q_E)$. To overcome this problem the following lemma is required.

Lemma 3.5.4. *Let V and W be semistable vector bundles over a curve C, such that $\mu(V) = \mu(W)$. Then:*

1. *if V is stable then every non-zero $f \in Hom(V, W) \cong H^0(C, V^\star \otimes W)$ is injective;*

2. *if W is stable then every non-zero $f \in Hom(V, W) \cong H^0(C, V^\star \otimes W)$ is surjective.*

Proof. Narasimhan-Ramanan's lemma is a corollary of Narasimhan-Seshadri [21] Proposition 4.4. The approach of Narasimhan-Seshadri is followed here.

We know that $H^0(C, V^\star \otimes W) \cong \mathrm{Hom}(V, W)$. Now choose $f \in \mathrm{Hom}(V, W)$. We have a factorisation $f = i \circ g \circ \eta$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V & \xrightarrow{\eta} & V_2 & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle g} & & \\ 0 & \longleftarrow & W_2 & \longleftarrow & W & \xleftarrow{i} & W_1 & \longleftarrow & 0 \end{array}$$

where g is of maximal rank; that is to say $\bigwedge^n g : \bigwedge^n V_2 \to \bigwedge^n W_1$ is non-zero; where $n = \mathrm{rank}(V_2) = \mathrm{rank}(W_1)$. Therefore $0 < \mathrm{rank}(V_2) = \mathrm{rank}(W_1) \leq \mathrm{rank}(W)$. The condition on g implies that $\deg(W_1) \geq \deg(V_2)$, hence $\mu(W_1) \geq \mu(V_2)$ because W_1 and V_2 have the same rank. The bundle V is semistable so $\mu(V_2) \geq \mu(V)$. Putting this together with the equality of slopes in the hypotheses gives the inequalities:

$$\mu(W_1) \geq \mu(V_2) \geq \mu(V) = \mu(W). \tag{3.59}$$

However, we are also assuming that W is semistable so $\mu(W_1) \leq \mu(W)$, this forces (3.59) to become:

$$\mu(W_1) = \mu(V_2) = \mu(V) = \mu(W) \tag{3.60}$$

In the first case of the lemma when V stable we will have a contradiction if $\mu(V_1) = \mu(V)$ and V_1 is a proper subbundle, so it must be the case that $\eta : V \tilde{\to} V_2$. The equality $\mu(W_1) = \mu(V_2)$ gives us that $\deg(W_1) = \deg(V_2)$ since the ranks of these two bundles are the same. The map $g : V_2 \to W_1$ is of maximal rank with $\deg(W_1) = \deg(V_2)$ so g is an isomorphism. Consequently

$$f = i \circ g \circ \eta : V \tilde{\to} V_2 \tilde{\to} W_1 \hookrightarrow W,$$

so f is injective.

In the second case the only way to avoid a contradiction to W being stable is if $i : W_1 \tilde{\to} W$. Again g must be an isomorphism because $\deg(W_1) = \deg(V_2)$; then $f = i \circ g \circ \eta : V \twoheadrightarrow V_2 \tilde{\to} W_1 \tilde{\to} W$. Hence f is surjective. □

Lemma 3.5.5. *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2, K)$ stable and generically globally generated with $h^0(C, E) \geq 5$; then $\Phi^\star U^\star$ is stable.*

Proof. We have the exact sheaf sequence

$$0 \to \Phi^\star U^\star \to E \to \mathcal{F} \to 0. \tag{3.61}$$

Suppose that $\det(\Phi^\star U^\star) = K(-D)$, for some divisor D of degree d. Let L be a line subbundle of $\Phi^\star U^\star$. Let $L(D_0)$ be the line bundle generated by the image of L in E, where D_0 is the divisor of points on which the image vanishes. We impose stability conditions on $\Phi^\star U^\star$ by relating the degrees of line bundles $L \subset \Phi^\star U^\star$ and $L(D_0) \subset E$.

Suppose that L is a destabilising line subbundle of $\Phi^\star U^\star$, so

$$\deg(L) = \mu(L) \geq \mu(\Phi^\star U^\star) = \frac{2g - 2 - d}{2}. \tag{3.62}$$

Now $g - 1 = \mu(E) > \deg(L(D_0)) \geq \deg(L)$. This inequality will have the mildest constraint on $\deg(L)$ if $D_0 = 0$ which means $L(D_0) = L$. Under this assumption $\deg(L) \leq g - 2$.

We know that $0 \to L \to \Phi^\star U^\star \to KL^{-1}(-D) \to 0$ and $h^0(C, \Phi^\star U^\star) = h^0(C, E)$ so we have:

$$h^0(C, E) = h^0(C, \Phi^\star U^\star) \leq h^0(C, L) + h^0(C, KL^{-1}(-D)). \tag{3.63}$$

We already have an upper bound on $\deg(L)$ which will allow us to control $h^0(C, L)$ using Brill-Noether arguments. Now look at $\deg(KL^{-1}(-D))$. From equation (3.62) we have $-d \leq 2\deg(L) - 2g + 2$. Therefore:

$$\deg(KL^{-1}(-D)) = 2g - 2 - \deg(L) - d \leq \deg(L) \leq g - 2. \tag{3.64}$$

We now find an upper bound on $h^0(C, L)$ and $h^0(C, KL^{-1}(-D))$ by calculating the relevant Brill-Noether number:

$$\rho(r, g-2) = g - (r+1)(g - (g-2) + r) = g - r^2 - 3r - 2. \tag{3.65}$$

For a generic curve this number is nonnegative, so taking $g \leq 11$ tells us that $r \leq 1$. Substituting this value into equation (3.63) gives:

$$h^0(C, E) = h^0(C, \Phi^\star U^\star) \leq H^0(C, L) + h^0(C, KL^{-1}(D)) \leq 4, \tag{3.66}$$

which contradicts our assumption that $h^0(C, E) \geq 5$. □

The following proposition will be proved in Chapter 4 subject to Assumption 4.2.1 .

Proposition 3.5.6. *Let C be a generic curve of genus $g \leq 11$. Take $E \in \mathcal{SU}(2, K)$ a stable generically globally generated bundle with $h^0(C, E) \geq 5$. Then Q_E is semistable.*

In order to show that $\bigwedge^i Q_E$ is semistable we introduce some notation used in Butler [5]. The Harder-Narasimhan filtration of a vector bundle F over a curve C is the unique filtration:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_s = F \tag{3.67}$$

where F_i/F_{i-1} is semistable and $\mu_i(F) = \mu(F_i/F_{i-1})$ is a strictly decreasing function of i. We define:

$$\mu^-(F) = \mu_s(F) = \mu(F_s/F_{s-1})$$
$$\mu^+(F) = \mu_1(F) = \mu(F_1).$$

We also have the following equivalent definition:

$$\mu^-(F) = \min\{\mu(Q) \mid F \to Q \to 0\}$$
$$\mu^+(F) = \max\{\mu(S) \mid 0 \to S \to F\}.$$

Note that S and Q need not be proper subbundles and quotient bundles. We have $\mu^+(F) \geq \mu(F) \geq \mu^-(F)$ with equality if and only if F is semistable.

In Butler [5] Lemma 2.5 it is shown that for F a vector bundle over a curve C that:

$$\mu^+(\bigwedge^i F) \leq i\mu^+(F)$$
$$\mu^-(\bigwedge^i F) \geq i\mu^-(F).$$

Now return to our consideration of $\bigwedge^i Q_E$. We know that Q_E is semistable so $\mu^+(Q_E) = \mu^-(Q_E)$, therefore:

$$\mu^+(\bigwedge^i Q_E) \leq i\mu^+(Q_E) = i\mu^-(Q_E) \leq \mu^-(\bigwedge^i Q_E). \tag{3.68}$$

In general $\mu^+(\bigwedge^i Q_E) \geq \mu^-(\bigwedge^i Q_E)$, so equation (3.68) gives us equality and hence semistability of $\bigwedge^i Q_E$. We have proved the following result.

Proposition 3.5.7. *Let C be a generic curve of genus $g < 12$ and $E \in \mathcal{SU}(2, K)$ stable and generically globally generated with $h^0(C, E) \geq 5$. Then $\bigwedge^i Q_E$ is semistable.*

The corollary to the following lemma computes the slopes of the $\bigwedge^i Q_E$ which we need if we are to use Lemmas 3.5.3 and 3.5.4.

Lemma 3.5.8. *Let V be a rank n vector bundle over a curve C, then:*

$$deg(\bigwedge^i V) = \binom{n-1}{i-1} deg(V).$$

Proof. By the splitting principle, Chern classes calculated for a split bundle are the same as those of a nonsplit bundle. Suppose that $V = \bigoplus_{s=1}^n L_s$, where $c_1(L_s) = \alpha_s$.

Then the Chern polynomial of V is:

$$c_t(V) = \prod_{s=1}^{n}(1+\alpha_s t).$$

Assuming that V is the direct sum of line bundles L_s means that we can expand the exterior powers of V:

$$\bigwedge^i V = \bigwedge^i(\bigoplus_{s=1}^{n} L_s) = \bigoplus_{s_1<\cdots<s_i} (L_{s_1}\otimes\cdots\otimes L_{s_i}). \tag{3.69}$$

For vector bundles W_1 and W_2, $c_t(W_1 \oplus W_2) = c_t(W_1).c_t(W_2)$ (Witney Product formula) and $c_t(W_1 \otimes W_2) = \text{rank}(W_2)c_t(W_1) + \text{rank}(W_1)c_t(W_2)$. We can now deduce from equation (3.69) that the Chern polynomial of $\wedge^i V$ is:

$$c_t(\bigwedge^i V) = \prod_{s_1<\cdots<s_i} (1+(\alpha_{s_1}+\cdots+\alpha_{s_i})t).$$

To calculate $c_1(\bigwedge^i V)$ we would like to determine how many times $(\alpha_1+\cdots+\alpha_n)t$ occurs in the above sum, as $\alpha_1+\cdots+\alpha_n = \deg(V)$. However, $c_1(\bigwedge^i V)$ is symmetric in the α_s so it will be sufficient to count the appearances of α_1. For each α_1 a further $i-1$ of the α_s must be chosen from $\{\alpha_2,\ldots,\alpha_{n-1}\}$ to complete every summand containing α_1. Hence the first Chern class is: $c_1(\bigwedge^i V) = \binom{n-1}{i-1}\deg(V)$. $\square$

To ease the notation let $h^0 = h^0(C,E)$.

Corollary 3.5.9. *Let $d = deg(D)$ where $K(-D) = det(\Phi_E^\star U^\star)$; then*

$$\mu(\bigwedge^i Q_E) = \frac{i}{h^0-2}(2g-2-d).$$

Proof. By the definition of Q_E there is an exact sequence on C:

$$0 \to \Phi_E^\star U \to H^0(C,E)^\star \otimes \mathcal{O} \to Q_E \to 0.$$

This gives information about determinants:

$$\mathcal{O} = \det(H^0(C,E)^\star\otimes\mathcal{O}) = \det(\Phi_E^\star U)\otimes\det(Q_E) = K^{-1}(D)\otimes\det(Q_E).$$

Therefore $\det(Q_E) = K(-D)$, which has degree $2g-2-d$. By Lemma 3.5.8 we have $\deg(\bigwedge^i Q_E) = \binom{h^0-3}{i-1}\deg(Q_E) = \binom{h^0-3}{i-1}(2g-2-d)$. From this we determine the slope:

$$\begin{aligned}\mu(\bigwedge^i Q_E) &= \frac{\deg(\bigwedge^i Q_E)}{\operatorname{rank}(\bigwedge^i Q_E)} = \frac{\binom{h^0-3}{i-1}}{\binom{h^0-2}{i}}(2g-2-d)\\ &= \frac{(h^0-3)!}{(i-1)!(h^0-2-i)!} \times \frac{i!(h^0-2-i)!}{(h^0-2)!}(2g-2-d)\\ &= i\frac{2g-2-d}{h^0-2}.\end{aligned}$$

□

With the results describing the kernel bundle in place we are now in a position to complete the proof of Proposition 3.2.4.

Consider the case $k=1$, where $h^0(C,E) = 5,6$. From equation (3.58) we have that

$$h^0(C,B_2) \le h^0(C,K^{-1}(D)) + h^0(C,\Phi_E^\star U \otimes Q_E).$$

We would like to prove that $h^0(C,B_2) = 0$ by showing that the upper bound is zero.

From Lemma 3.5.5 we know that $\Phi_E^\star U^\star$ is stable and from Proposition 3.5.6 that Q_E is semistable. If $\mu(\Phi_E^\star U^\star) > \mu(Q_E)$ then the hypotheses of Lemma 3.5.3 are satisfied and $h^0(C,\Phi_E^\star U \otimes Q_E) = 0$. By Corollary 3.5.9 the slopes of Q_E for $h^0(C,E) = 5,6$ are given by:

$$\mu(Q_E) = \begin{cases} \frac{1}{3}(2g-2-d) \text{ if } h^0(C,E) = 5\\ \frac{1}{4}(2g-2-d) \text{ if } h^0(C,E) = 6.\end{cases}$$

However, $\mu(\Phi_E^\star U^\star) = \frac{1}{2}(2g-2-d)$.

We now show that $h^0(C,K^{-1}(D)) = 0$. If $h^0(K^{-1}(D)) > 0$ then we would have $\deg(K(-D)) \le 0$. The bundle $\Phi_E^\star U^\star$ is stable so for any subbundle L:

$$\deg(L) < \mu(\Phi_E^\star U^\star) = \frac{1}{2}\deg(K(-D)) \le 0.$$

Therefore $h^0(C,\Phi_E^\star U^\star) = \dim(H^0(C,\mathcal{O}\otimes\Phi_E^\star U^\star)) = \dim(\operatorname{Hom}(\mathcal{O},\Phi_E^\star U^\star)) = 0$, a contradiction. Hence $h^0(C,B_2) = 0$, concluding the proof for $k=1$.

For $k = 2$ our bound 3.49 tells us that $h^0(C, E) = 6$. The situation is more subtle then for lower k. We are looking at the bound:

$$h^0(C, B_3) \leq h^0(C, K^{-1}(D) \otimes Q_E) + h^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E).$$

It is clear from Lemma 3.5.3 that $h^0(C, K^{-1}(D) \otimes Q_E) = 0$ because $\mu(K(-D)) = 2g - 2 - d > \frac{1}{4}(2g - 2 - d) = \mu(Q_E)$, and both $K(-D)$ and Q_E are semistable. On the other hand, although both $\Phi_E^\star U^\star$ and $\bigwedge^2 Q_E$ are semistable we have $\mu(\Phi_E^\star U^\star) = \frac{1}{2}(2g - 2 - d) = \mu(\bigwedge^2 Q_E)$, so we cannot use Lemma 3.5.3 to show that $H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E) = 0$. However, the hypotheses of Lemma 3.5.4 are satisfied telling us that $f \in \mathrm{Hom}(\Phi_E^\star U^\star, \bigwedge^2 Q_E) \cong H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E)$ is injective. This fact is central to the proof of the following lemma.

Lemma 3.5.10. *Let C be a generic curve of genus $g \leq 11$ and $E \in \mathcal{SU}(2, K)$ a stable generically globally generated vector bundle over C with $h^0(C, E) = 6$. A decomposable vector of $\bigwedge^3 H^0(C, E)^\star$ is not in the image of $H^0(C, B_3) \subset \bigwedge^3 H^0(C, E)^\star$.*

Recall that the decomposable 3-vector $\pi \in \bigwedge^3 H^0(C, E)^\star$ represents the 3-plane V; where $\mathbb{P}(V)$ is the singular locus of our quadric q. We showed that the condition $\Phi_E(C) \subset \mathcal{V}_E$ forced π to lie in the image of $H^0(C, B_3)$ in $\bigwedge^3 H^0(C, E)^\star$. Therefore this lemma finishes the proof of Proposition 3.2.4.

Proof. Let j be the injection $j : B_3 \hookrightarrow \bigwedge^3 H^0(C, E)^\star \otimes \mathcal{O}$ and j' the induced map on sections $j' : H^0(C, B_3) \hookrightarrow \bigwedge^3 H^0(C, E)^\star$. By Lemma 3.5.2 we have an exact sequence:

$$0 \to K^{-1}(D) \otimes Q_E \xrightarrow{\phi_3} B_3 \xrightarrow{\psi_3} \Phi_E^\star U \otimes \bigwedge^2 Q_E \to 0. \qquad (3.70)$$

The cohomology sequence associated with (3.70) is:

$$0 \to H^0(K^{-1}(D) \otimes Q_E) \to H^0(C, B_3) \xrightarrow{\Psi_3} H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E) \to \dots$$

However $H^0(C, K^{-1}(D) \otimes Q_E) = 0$ so $\Psi_3 : H^0(C, B_3) \hookrightarrow H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E)$.

Suppose for a contradiction that $j'(s) = p_1 \wedge p_2 \wedge p_3 \in \bigwedge^3 H^0(C,E)^\star$ for some $s \in H^0(C, B_3)$. Also define $\sigma := \Psi_3(s)$. We know by Lemma 3.5.4 that:

$$H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E) \cong \{f : \Phi_E^\star U^\star \to \bigwedge^2 Q_E \mid f \text{ injective}\},$$

where the isomorphism is defined on the fibres by the vector space isomorphism:

$$\left(\Phi_E^\star U \otimes \bigwedge^2 Q_E\right)_x \cong \mathrm{Hom}\left((\Phi_E^\star U^\star)_x, (\bigwedge^2 Q_E)_x\right).$$

If the map $f \in \mathrm{Hom}(\Phi_E^\star U^\star, \bigwedge^2 Q_E)$ is injective then the restriction to the fibre f_x has rank 2.

We have $\sigma \in H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E)$, so $\sigma \cong f$ where $f : \Phi_E^\star U^\star \to \bigwedge^2 Q_E$ is injective. To obtain a contradiction we pick a point $x \in C$ and study $s(x)$ and $\sigma(x)$, to show that f_x has nonzero kernel.

At $x \in C$ we identify $(B_3)_x$ with a subspace of $\left(\bigwedge^3 H^0(C,E)^\star\right)_x$. Since we have that $j'(s) = p_1 \wedge p_2 \wedge p_3$ then $s(x) = p_1 \wedge p_2 \wedge p_3$. However, $s(x) \in (B_3)_x$ so $\bigwedge^3 \gamma(s(x)) = 0$, we therefore have:

$$\begin{aligned}
\bigwedge^3 \gamma(p_1 \wedge p_2 \wedge p_3) = 0 &\Leftrightarrow \gamma(p_1) \wedge \gamma(p_2) \wedge \gamma(p_3) = 0 \\
&\Leftrightarrow \{\gamma(p_1), \gamma(p_2), \gamma(p_3)\} \text{ is linearly dependent} \\
&\Leftrightarrow \sum_{i=1}^{3} \lambda_i \gamma(p_i) = 0 \text{ for } (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 - \{0\} \\
&\Leftrightarrow \sum_{i=1}^{3} \lambda_i p_i \in (\Phi_E^\star U)_x \text{ for } (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 - \{0\}.
\end{aligned}$$

Let $e^\star \in (\Phi_E^\star U)_x$ such that $e^\star = \sum_{i=1}^3 \lambda_i p_i$. Without loss of generality assume that $\lambda_1 \neq 0$, then:

$$e^\star \wedge p_2 \wedge p_3 = \left(\sum_{i=1}^{3} \lambda_i p_i\right) \wedge p_2 \wedge p_3 = \sum_{i=1}^{3} \lambda_i p_i \wedge p_2 \wedge p_3 = \lambda_1 p_1 \wedge p_2 \wedge p_3.$$

Thus by picking appropriate rescaled vectors ($P_2 = p_2$ and $P_3 = \frac{1}{\lambda_1} p_3$) we can say that $p_1 \wedge p_2 \wedge p_3 = e^\star \wedge P_2 \wedge P_3$.

We now have a good description of $s(x)$, from which we will be able to deduce information about $\sigma(x) := \Psi_3(s)(x)$. To do this we need to describe the map $B_3 \twoheadrightarrow \Phi_E^\star U \otimes \bigwedge^2 Q_E$ fibrewise. By Lemma 3.5.2 we have:

$$\begin{array}{ccc}
\Phi_E^\star U \otimes (\bigwedge^2 H^0(C,E)^\star \otimes \mathcal{O}) & \xrightarrow{1\otimes\bigwedge^2\gamma} & \Phi_E^\star U \otimes \bigwedge^2 Q_E \\
{\scriptstyle \beta\wedge 1}\big\downarrow & & \big\downarrow{\scriptstyle I} \\
B_3 & \xrightarrow{q} & B_3/\bigwedge^2 \Phi_E^\star U \otimes Q_E
\end{array}$$

where I is the isomorphism induced by the diagram. For vectors $\epsilon^\star \in (\Phi_E^\star U)_x$ and $v_2, v_3 \in H^0(C,E)$ we have:

$$I_x : \epsilon^\star \otimes [v_2]\wedge[v_3] \xrightarrow{(1\otimes\bigwedge^2\gamma)^{-1}} \epsilon^\star \otimes (v_2 \wedge v_3 + B_2) \xrightarrow{\beta\wedge 1} \epsilon^\star \wedge v_2 \wedge v_3 \xrightarrow{q} \epsilon^\star \wedge v_2 \wedge v_3 + \wedge^2 \Phi_E^\star U \otimes Q_E.$$

Therefore

$$\Phi_3 : s(x) = e^\star \wedge P_2 \wedge P_3 \xrightarrow{q} e^\star \wedge P_2 \wedge P_3 \xrightarrow{I^{-1}} e^\star \otimes [P_2] \wedge [P_3] = \sigma(x). \tag{3.71}$$

Let us now return to $f \in \operatorname{Hom}(\Phi_E^\star U^\star, \bigwedge^2 Q_E)$, the injective map associated to $\sigma \in H^0(C, \Phi_E^\star U \otimes \bigwedge^2 Q_E)$. Then at the point x we have $\sigma(x) = e^\star \otimes [P_2] \wedge [P_3] \cong f_x \in \operatorname{Hom}\left((\Phi_E^\star U^\star)_x, (\bigwedge^2 Q_E)_x\right)$. The isomorphism is given by:

$$f_x(v) = e^\star(v)[P_2] \wedge [P_3] \text{ for all } v \in (E)_x.$$

Consequently $\operatorname{im}(f_x) = \langle [P_2] \wedge [P_3] \rangle \subset (\bigwedge^2 Q_E)_x$, so in particular f_x has rank 1 which contradicts f being an injection. $\square$

Chapter 4

Stability of a kernel bundle

In this chapter we prove the following proposition subject to an assumption (Assumption 4.2.1).

Proposition 3.5.6 *Let C be a generic curve of genus $g \leq 11$. Take $E \in \mathcal{SU}(2, K)$ a stable generically globally generated bundle with $h^0(C, E) \geq 5$. Then Q_E is semistable.*

To prove this result in section 4.1 we adapt the work of Butler [5] where he describes the stability of the kernel bundle M_E of a surjective evaluation map:

$$0 \to M_E \to H^0(C, E) \otimes \mathcal{O} \xrightarrow{e} E \to 0.$$

Our interest is in Q_E which at first sight is the dual of M_E. However, the relationship is more subtle than this as E is only assumed to be generically globally generated. In section 4.2 we build on these general results to complete the proof of Proposition 3.5.6.

4.1 Preliminaries

To start we define the bundle Q_E. Suppose that E is a generically globally generated vector bundle of rank r, so the evaluation map $H^0(C, E) \otimes \mathcal{O} \to E$ is surjective for all but a finite number of points of the curve. Define D to be those points of the curve for which this map drops rank. From the evaluation map we induce a morphism $\Phi_E : C \to Gr(r, H^0(C,E)^\star)$. On the Grassmannian we have the tautological sequence $0 \to U \to H^0(C,E)^\star \otimes \mathcal{O} \to Q \to 0$ which we then pull back to C with Φ_E. This gives the sequence $0 \to \Phi_E^\star U \to H^0(C,E)^\star \otimes \mathcal{O} \to Q_E \to 0$, implicitly defining the bundle Q_E.

Now dualise this sequence to define M_E:

$$0 \to M_E \to H^0(C,E) \otimes \mathcal{O} \xrightarrow{\beta} \Phi_E^\star U^\star \to 0. \tag{4.1}$$

By the assumption that E is generically globally generated we know that β is the evaluation map on $C - D$.

In Butler's treatment of M_E he assumes that E is globally generated or in other words that β is the evaluation map. In the following lemma we modify Butler's Lemma 1.9; to deal with M_E as defined in equation (4.1). Note here that Q_E is semistable if and only if M_E is semistable.

Lemma 4.1.1. *Suppose E is a vector bundle over a curve C which is generically globally generated. Let N be a stable subbundle of M_E with maximal slope. There exists a vector bundle F with $\mu(F) \leq \mu^+(\Phi_E^\star U^\star)$; and a vector space $V \subset H^0(C,F)$ such that:*

$$0 \to N \to V \otimes \mathcal{O} \to F \to 0.$$

Proof. The vector subspace V is constructed first and then the vector bundle F. Clearly $N \hookrightarrow M_E$ since N is a subbundle of M_E. By our definition $M_E \hookrightarrow H^0(C,E) \otimes \mathcal{O}$; taking the composition we obtain the injective map $N \to M_E \to H^0(C,E) \otimes \mathcal{O}$. We dualise to get $H^0(C,E)^\star \otimes \mathcal{O} \to M_E^\star \to N^\star$, which induces a map on sections $j : H^0(C,E)^\star \to H^0(C,N^\star)$. Let $V^\star$ be defined to be $\mathrm{im}(j)$.

We now turn to the bundle F. The map $V^\star \otimes \mathcal{O} \to N^\star$ is defined by restriction of $H^0(C,E)^\star \otimes \mathcal{O} \to N^\star$ and is therefore surjective. The vector bundle $F^\star$ is defined to be the kernel of the map j, and so fits into the exact sequence:

$$0 \to F^\star \to V^\star \otimes \mathcal{O} \to N^\star \to 0. \tag{4.2}$$

By considering the short exact sequences (4.1) and the dual of (4.2), and noting that $j^\star : V \hookrightarrow H^0(C, E)$ (because $j : H^0(C, E)^\star \twoheadrightarrow V$) we form the following useful commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & V \otimes \mathcal{O} \arrow[r, "\delta"] \arrow[d, "\gamma"'] & F \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M_E \arrow[r] & H^0(C,E) \otimes \mathcal{O} \arrow[r, "\beta"] & \Phi_E^\star U^\star \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

where α is the induced map. Note that δ and β are both defined to be the evaluation map away from D.

We now show that $\alpha \neq 0$; for a contradiction assume the converse. Pick a point $v \in V$; then:

$$0 = \alpha(\delta(\{v\} \otimes \mathcal{O})) = \beta(\gamma(\{v\} \otimes \mathcal{O})).$$

However $\gamma(\{v\} \otimes \mathcal{O}) = \{s\} \otimes \mathcal{O}$ for some $s \in H^0(C, E)$. Thus

$$\beta(\{s\} \otimes \mathcal{O}) = 0 \Rightarrow s(p) = 0 \text{ for } p \in C - D.$$

Consequently $s = 0$; and so $v = 0$ because γ is injective. This means that $V = 0$; a contradiction because $N \hookrightarrow V \otimes \mathcal{O}$.

Now that F and V have been defined we show their properties enumerated in Lemma 4.1.1. We start by proving that $V \subset H^0(C, F)$. Consider the bottom line of diagram (4.3) and take the long exact sequence in cohomology:

$$0 \to H^0(C, M_E) \to H^0(C, E) \xrightarrow{B} H^0(C, \Phi_E^\star U^\star) \to \ldots. \tag{4.4}$$

Suppose that $B(s) = 0$, then $0 = B(s)(x) = \beta(s(x))$ for all $x \in C$. Away from D the map β coincides with the evaluation map, so $0 = \beta(s(x)) = s(x)$ for all $x \in C - D$. Therefore s must be zero for all points of the curve so we conclude that $\ker(B) = 0$.

By sequence (4.4) the kernel of B is just $H^0(C, M_E)$; however N is a subbundle of M_E which means that $H^0(C,N) \hookrightarrow H^0(C,M_E) = 0$, therefore $H^0(C,N) = 0$. Now consider the top line of diagram (4.3) and take the cohomology sequence to get:

$$0 \to H^0(N) \to V \to H^0(F) \to \dots$$

However $H^0(C,N) = 0$ so $V \hookrightarrow H^0(C,F)$ as required.

The conditions on the slopes of F and E are now discussed. Our first observation is that

$$V \overset{A}{\hookrightarrow} H^0(C, \alpha(F)), \tag{4.5}$$

where A is given by $A(v)(x) = (\alpha \circ \delta)(v(x))$ for v and x points in V and C respectively. To see the injectivity of A consider $v \in V$ such that $A(v) = 0$. Let $\mathcal{F}$ be the sheaf defined by v in the following way:

$$\langle v \rangle \otimes \mathcal{O} \xrightarrow{\delta} \mathcal{F}.$$

Now restrict attention to a point $x \in C$ and use diagram (4.3):

$$\begin{aligned}(\beta \circ \gamma)(v)(x) &= (\alpha \circ \delta)(v)(x) \\ &= A(v)(x) \\ &= 0.\end{aligned}$$

This holds true for all $x \in C$, therefore $\gamma(\langle v \rangle \otimes \mathcal{O})$ lies in the image of M_E in $H^0(C,E) \otimes \mathcal{O}$. However, $0 = H^0(C, M_E) \supset H^0(C, \langle v \rangle \otimes \mathcal{O}) \cong \mathbb{C}$ - a contradiction.

To obtain the conditions on the slopes of F and E the following diagram is constructed:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & N \arrow[r] \arrow[d] & V \otimes \mathcal{O} \arrow[r] \arrow[d, "A"'] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{\alpha(F)} \arrow[r] & H^0(C, \alpha(F)) \otimes \mathcal{O} \arrow[r] & \alpha(F) \arrow[r] \arrow[d] & 0 \\ & & & 0 & \end{tikzcd} \tag{4.6}$$

We now show that the induced map $N \to M_{\alpha(F)}$ is injective. The bundle N is mapped to zero in F, so the image of N in $V \otimes \mathcal{O}$ will also be zero in $\alpha(F)$ by commutativity of the right hand square. This means that $A(N)$ must lie in $M_{\alpha(F)} \subset H^0(C, \alpha(F)) \otimes \mathcal{O})$, but A is an injection so we must have $N \hookrightarrow M_{\alpha(F)}$.

We have $N \subset M_{\alpha(F)} \subset M_E$; and N has maximal slope in M_E so $\mu(N) \geq \mu(M_{\alpha(F)})$. The horizontal lines of (4.6) give that:

$$\mu(N) = \frac{-\deg(F)}{\operatorname{rank}(N)} \text{ and } \mu(M_{\alpha(F)}) = \frac{-\deg(\alpha(F))}{\operatorname{rank}(M_{\alpha(F)})}.$$

These expressions combined with the inequality between the slopes of N and $M_{\alpha(F)}$ gives:

$$\begin{aligned} \frac{-\deg(\alpha(F))}{\operatorname{rank}(M_{\alpha(F)})} = \mu(M_{\alpha(F)}) \leq \mu(N) = \frac{-\deg(F)}{\operatorname{rank}(N)} \\ \Rightarrow \deg(F)\frac{\operatorname{rank}(M_{\alpha(F)})}{\operatorname{rank}(N)} \leq \deg(\alpha(F)) \\ \Rightarrow \deg(F) \leq \deg(\alpha(F)). \end{aligned} \tag{4.7}$$

Moreover, it can be seen that $\deg(F) = \deg(\alpha(F))$ if and only if $N = M_{\alpha(F)}$. Clearly $\operatorname{rank}(F) \geq \operatorname{rank}(\alpha(F))$ so by the above condition on degrees:

$$\mu(F) \leq \mu(\alpha(F)) \leq \mu^+(\Phi_E^\star U^\star).$$

$\square$

In Proposition 1.4 Butler goes on to compute an upper bound for the slope of subbundles of M_E. With Butler's Lemma 1.9 reworked for generically globally generated bundles the proof of Lemma 1.4 holds for the following result.

Proposition 4.1.2. *Let E be a vector bundle over C generically globally generated by global sections, and if $sup_S\{\mu(S) \mid S$ proper subbundle of $\Phi_E^\star U^\star\} < 2g$ then*

$$sup_T\{\mu(T) \mid T \text{ proper subbundle of } M_E\} \leq -2.$$

4.2 The Stability of M_E

In this section we use the general results from section 4.1 and one additional assumption to show that M_E is semistable. We achieve this by first noting in Corollary 4.2.3 that for curves with $W^2_{g-2} = \emptyset$ both F and N must be line bundles. By Brill-Noether considerations F and N are line bundles for generic curves of genus $g < 12$. It is then shown in Proposition 4.2.4 that if we have N a line bundle, $H^0(C, F^\star) = 0$ and certain constraints on the linear series on C then M_E is semistable. Noting that Q_E is semistable if and only if M_E is semistable it is then shown that Proposition 3.5.6 is a corollary of Proposition 4.2.4.

We now consider the map $\alpha : F \to \Phi^\star_E U^\star$, defined in the commutative diagram (4.3). The map α is non-zero, so $\alpha(F)$ is a subsheaf of $\Phi^\star_E U^\star$. To continue our proof we make the following assumption.

Assumption 4.2.1. *The subsheaf $\alpha(F)$ is a line subbundle of $\Phi^\star_E U^\star$.*

If $M_E = N$ then we have M_E stable and our proof is complete, so from now on we assume that N is a proper subbundle of M_E.

Knowing that $\mathrm{rank}(\alpha(F)) = 1$ means that Brill-Noether analysis may be used to find a relationship between the genus of the curve and the ranks of the bundles F and N. First denote the ranks of F and N by f and n respectively. We note that $n \geq 1$, as the bundle M_E will certainly contain a stable subbundle of maximal rank. The bundle F must have rank at least one also, as otherwise $N \cong V \otimes \mathcal{O}$ - the trivial bundle, with slope zero - and Proposition 4.1.2 states that $\mu(N) \leq -2$. We have;

$$\mathrm{rank}(F),\ \mathrm{rank}(N) \geq 1. \tag{4.8}$$

Lemma 4.2.2. *Let $E \in \mathcal{SU}(2, K)$ be a stable generically globally generated bundle with $h^0(C, E) \geq 5$. If rank$(F) = f$ and rank$(N) = n$ then C must have a g^r_d where $r \geq n + f - 1$ and $d \leq g - 2$. In particular if C is a generic curve with then rank$(F) = f$ and rank$(N) = n$ then $g \geq (f + n)(f + n + 1)$.*

Proof. To obtain this bound the dimension of V is expressed in terms of f and n. Since $0 \to N \to V \otimes \mathcal{O} \to F \to 0$ then $\dim(V) = n + f$. In equation (4.5) it was shown that $V \hookrightarrow H^0(C, \alpha(F))$. Hence $r(\alpha(F)) \geq \dim(V) - 1 = f + n - 1$. However, $\alpha(F) \subset \Phi_E^\star U^\star$, where $\Phi_E^\star U^\star$ is stable by Lemma 3.5.5. Moreover, $\mu(\Phi_E^\star U^\star) \leq \mu(E)$, with equality when E is globally generated. Hence $\deg(\alpha(F)) \leq g - 2$. Therefore $\alpha(F) \in W_d^r$, where $r \geq f+n-1$ and $d \leq g-2$. For a generic curve the Brill-Noether number of $\alpha(F)$ is at least zero; this gives:

$$0 \leq \rho(\alpha(F)) \leq \rho(f+n-1, g-2) = g - (f+n)(f+n+1) \Rightarrow g \geq (f+n)(f+n+1).$$

□

From this we can deduce the Corollary that will be used in showing M_E is semistable.

Corollary 4.2.3. *Let $E \in \mathcal{SU}(2, K)$ be a stable generically globally generated bundle with $h^0(C, E) \geq 5$. If the curve C has no g_{g-2}^2 then $n = f = 1$. Furthermore, any generic curve of genus $g < 12$ has $n = f = 1$.*

Proof. We know (by (4.8)) that $n, f \geq 1$. If one f or n is greater than 1 then by Lemma 4.2.2 we must have a g_d^r where $r \geq 2$ and $d \leq g - 2$ - contradicting our hypotheses. If the curve is generic then again by Lemma 4.2.2, if one of f and n is greater than 1 then the genus of the curve must be at least $(f+n)(f+n+1) \geq 12$. □

We are now in a position to prove our principal result of the section.

Proposition 4.2.4. *Let $E \in \mathcal{SU}(2, K)$ be stable generically globally generated bundle with $h^0(C, E) \geq 5$. Let N be the maximal stable subbundle of M_E and F defined by*

$$0 \to N \to V \otimes \mathcal{O} \to F \to 0,$$

where $V \subset H^0(C, F)$. Assume that N is a line bundle and $H^0(C, F^\star) = 0$, then M_E is semistable if $G^1_{\frac{2g-3}{3}} = \emptyset$.

Proof. Let v denote the dimension of V. It is the case that $v \geq 2$, since $v = n + f$ and as was seen in (4.8) $n, f \geq 1$.

We are assuming that N is line bundle, so that we can use Brill-Noether arguments. However it is necessary to have a positive line bundle so we look at $N^\star$ because Theorem 4.1.2 tells us that:

$$\deg(N^\star) = -\deg(N) = -\mu(N) \geq 2.$$

We would now like to find a decent lower bound on $r(N^\star)$; to do this look at the sequence:

$$0 \to F^\star \to V^\star \otimes \mathcal{O} \to N^\star \to 0,$$

and take the associated cohomology sequence:

$$0 \to H^0(C, F^\star) \to V^\star \to H^0(C, N^\star) \to \ldots$$

By the hypothesis that $H^0(C, F^\star) = 0$ we have $V^\star \hookrightarrow H^0(C, N^\star)$ and thus $h^0(C, N^\star) \geq v \geq 2$.

If we assume for a contradiction that M_E is not semistable then we must have $\mu(N) > \mu(M_E)$ or equivalently:

$$\deg(N^\star) = \mu(N^\star) < \mu(M_E^\star) \leq \frac{2g-2}{h^0(E)-2}.$$

To obtain a bound on $\deg(N^\star)$ in terms of g we look at the highest value that $\mu(M_E^\star)$ can take when $h^0(C, E)$ is varied. This value is $\mu(M_E^\star) = [\frac{2}{3}(g-1)]$ which is attained when $h^0(C, E) = 5$. Consequently the bound will depend on $g(\text{mod}3)$. By taking the largest value - when $g = 0(\text{mod}3)$ - we have:

$$\deg(N^\star) \leq \frac{2g-3}{3}. \tag{4.9}$$

□

Proof of Proposition 3.5.6 It will be sufficient to show that the hypotheses of Proposition 4.2.4 are met. With Assumption 4.2.1 and the curve being generic of genus

$g \leq 11$ the Corollary 4.2.3 holds; consequently N and F are line bundles. Consider the exact sequence

$$0 \to F^\star \to V^\star \otimes \mathcal{O} \to N^\star \to 0,$$

we know that $\deg(F^\star) = -\deg(N^\star) \leq -2$. Hence $F^\star$ is a line bundle of negative degree so we must have $H^0(C, F^\star) = 0$. Now consider a linear series in $G^1_{\frac{2g-3}{3}}$, we have:

$$\rho(1, \frac{2g-3}{3}) = g - (2)(g - \frac{2g-3}{3} + 1) = \frac{g}{3} - 4. \tag{4.10}$$

The curve is generic of genus $g \leq 11$; so we must have $G^1_{\frac{2g-3}{3}} = \emptyset$.

Chapter 5

Brill-Noether locus of genus 9 curves

In Mukai [19] it was stated without proof that the Brill-Noether locus $\overline{\mathcal{W}^5}$ of a generic curve of genus 9 is a singular quartic 3-fold. It is the aim of this chapter to study some properties of this example. We will be assuming that $\overline{\mathcal{W}^r}$ is an irreducible variety.

It is first shown that points of $\overline{\mathcal{W}^5}$ are generically extensions of line bundles $\mathcal{O}(D)$ for $D \in S^4C$. A map $\phi : S^4C \dashrightarrow \overline{\mathcal{W}^5}$ is then defined by $D \mapsto \epsilon_D(\Omega^0_D)$. The space S^4C is given a determinantal description which enables us to naturally define a filtration $S^4C = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \emptyset$, where

$$\Sigma_1 = \{D \in S^4C \mid \dim(\Omega^0_D) = 1\}.$$

In section 5.2 we look at the condition $\Sigma_2 = \emptyset$. Assuming that $\Sigma_2 = \emptyset$ we use the determinantal structure on S^4C to take the canonical blowup of Σ_1 and obtain a map $\widetilde{\phi} : \widetilde{S^4C} \dashrightarrow \overline{\mathcal{W}^5}$.

In section 5.3 the class of Σ_1 is calculated under the assumption that it has the "expected" dimension derived from our determinantal description.

The points of $\widetilde{S^4C}$ for which $\widetilde{\phi}$ is undefined are characterised. An expression is then given that determines the degree of $\overline{\mathcal{W}^5}$ in terms of the class of the general fibre and the pull-back of the hyperplane bundle.

5.1 Maps to $\overline{\mathcal{W}^5}$

To describe this Brill-Noether locus we consider the extensions of divisors D that are mapped to $\overline{\mathcal{W}^5} \subset \mathcal{SU}(2,K)$ by the moduli map ϵ_D. Supposing that $E \in \mathbb{P}\mathrm{Ext}^1(K-D,D)$ for a divisor D, we have equation (2.5):

$$h^0(C,E) = g + 1 - \mathrm{Cliff}(D) - n \text{ for } E \in \Omega_D^n - \Omega_D^{n-1}.$$

However $\mathrm{Cliff}(D) \geq \mathrm{Cliff}(C) = [\frac{g-1}{2}] = 4$ for a generic curve of genus 9 (see (2.2)). It is required that $h^0(C,E) = 6$ so the above equation gives us that $n = 0$ and $\mathrm{Cliff}(D) = 4$. The extensions that we are interested in will therefore be $E \in \Omega_D^0 \subset \mathbb{P}\mathrm{Ext}^1(K-D,D)$ where $\mathcal{O}(D) \in W_4, W_6^1$ or W_8^2.

We would like to show that bundles in $\overline{\mathcal{W}^5}$ are generically extensions of $D \in S^4C$.

First we consider the extensions of D for $\mathcal{O}(D) \in W_8^2$. All such extensions are S-equivalent to the semistable bundle $\mathcal{O}(D) \oplus K(-D) \in \mathcal{SU}(2,K)$. Moreover, we have that $\rho(2,8) = 0$, so W_8^2 is a finite number of points. In fact Castelnuovo's Theorem 1.1.9 tells us there are precisely 42 such bundles, which in occur in Serre dual pairs. Therefore there are only 21 points in $\overline{\mathcal{W}^5}$ that represent extensions of these divisors. Moreover, by condition 1.11 we know that $\mathcal{W}^5$ is smooth, so the 21 semistable bundles constitute the singular locus of Mukai's description.

Now consider extensions of D for $\mathcal{O}(D) \in W_6^1$. We prove that the map:

$$\psi : W_6^1 \dashrightarrow \overline{\mathcal{W}^5} \text{ where } \psi : \mathcal{O}(D) \mapsto \epsilon_D(\Omega_D^0) \tag{5.1}$$

has image which is at most 2 dimensional. Recall from chapter 2 that for $D \in S^6C$

with $r(D) = 1$ we have the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & \mathbb{P}H^1(C,D) \cong \mathbb{P}^3 \arrow[d, "\mathrm{Ver}"] \\ \mathbb{P}^{11} \cong \mathbb{P}\mathrm{Ext}^1(K-D,D) \arrow[r, "\delta"] & \mathbb{P}\mathrm{Sym}^2 H^1(C,D) \cong \mathbb{P}^9. \end{tikzcd}$$

Clearly $\dim(\ker(\delta_D)) \geq 2$, with equality if and only if δ_D surjective. If $\dim(\ker(\delta_D)) = 2$ then $\Omega_D^0 \cong \mathbb{P}^1$ and $\epsilon_D(\Omega_D^0)$ is at most one dimensional. We know that $\dim(W_6^1) = 1$ so if δ_D were surjective for all $D \in S^6C$ with $r(D) = 1$ then $\dim(\psi(W_6^1)) \leq 2$.

The map δ_D is surjective if and only if $\lambda_{|K-D|}(C)$ does not lie in a quadric in $\mathbb{P}^3$. This condition is shown in Lemma 5.1.2, the proof of which requires the following result on the singularities of $\lambda_{|K-D|}(C)$.

Lemma 5.1.1. *Let C be a generic curve of genus 9 and $\lambda : C \xrightarrow{|K-D|} \mathbb{P}^3$, where $D \in S^6C$ with $r(D) = 1$. Then $\lambda(C)$ is either smooth or has singularities of multiplicity 2.*

Proof. Note to start that λ is birational by Lemma 2.2.10. The curve $\lambda(C)$ may be regarded as the projection of C embedded in canonical space away from the divisor D. From now on C and its image in canonical space will be identified. If $\lambda(C)$ has a singularity of multiplicity d then there exists a divisor D' of degree d lying on C that is mapped to a point on projection from the divisor D. Thus the intersection of $\overline{D}$ and $\overline{D'}$ must be a hyperplane in $\overline{D'}$. By the geometric Riemann-Roch formula the span of D is 4 dimensional; this gives an upper bound of 5 on the dimension of $\overline{D'}$, which is attained when $\overline{D} \subset \overline{D'}$. The relationship between the spans of D and D' leads us to consider $D + D'$. The geometric Riemann-Roch formula gives:

$$5 = \dim(\overline{D + D'}) = (6 + d - 1) - r(D + D') \Rightarrow r(D + D') = d.$$

The curve C is generic so the Brill-Noether numbers of all divisors on C must be non-negative:

$$0 \leq \rho(D + D') \leq \rho(d, d+6) = 9 - (d+1)(9 - (d+6) + d) = 6 - 3d.$$

We conclude that $d \leq 2$, so the curve $\lambda(C)$ may only have double points. $\square$

The next lemma proves that the image of ψ is at most 2 dimensional.

Lemma 5.1.2. *Suppose C is a curve of genus 9 let $\lambda : C \to \mathbb{P}^3$ be given by the linear series $|K - D|$ where $D \in S^6C$ with $r(D) = 1$. The image $\lambda(C)$ cannot lie in a quadric.*

Proof. In the proof we follow Lemma 2.2.6. We know by Lemma 2.2.10 that λ is birational so in the following we will abuse notation by taking $C = \lambda(C)$.

Clearly the curve C cannot be mapped to a rank 1 or 2 quadric as this violates the nondegeneracy of λ.

A quadric of rank 3 is the quadric cone, which we will refer to as X. Consider the projection π down X onto the base conic. The degree of C is 10 so for the projection to occur C must pass through the vertex of X an even number of times (and may not meet the vertex at all). Assuming that C meets the vertex of the cone $2r$ times $(0 \leq r \leq 5)$ then the projection will map the curve $d:1$ onto the base conic where

$$d = \frac{1}{2}(10 - 2r) = 5 - r \text{ for } 0 \leq r \leq 5. \tag{5.2}$$

The linear series on C associated to $\pi|_C$ will be a g^1_d. However, the curve C is generic so the Brill-Noether number of this g^1_d must be non-negative:

$$0 \leq \rho(1, d) = 9 - (1 + 1)(9 - d + 1) \Rightarrow d \geq 6.$$

This contradicts (5.2) so C cannot lie on a rank 3 quadric.

The final case is when the curve maps to a smooth quadric which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that the image of the curve is $C = l_1E_0 + l_2B$. We know that $\deg(C) = 10 = l_1 + l_2$, therefore one of the l_i is less than 6. Projecting from the larger l_i gives a g^1_d, where $d < 6$; but a generic curve of genus 9 cannot have a linear series of this kind. This concludes the proof that C cannot lie on a quadric. □

We know that $\dim(\overline{\mathcal{W}^5}) = 3$ and we have shown that at most a 2 dimensional subset of $\overline{\mathcal{W}^5}$ comes from extensions of D, for $\mathcal{O}(D) \in W^1_6$ or $\mathcal{O}(D) \in W^2_8$. It must be the case that a generic point of $\overline{W^5}$ is the extension of $D \in S^4C$. Consequently we define the rational map

$$\phi : S^4C \dashrightarrow \overline{W^5} \text{ where } \phi : D \mapsto \epsilon_D \Omega_D^0. \tag{5.3}$$

From our discussion above the image of ϕ is 3 dimensional. Our aim is to perform a series of blow-ups on S^4C in order to construct a morphism $\Phi : Bl(S^4C) \to \overline{W^5}$. To do this we need a better understanding of ϕ which requires us to describe the extension spaces Ω_D^0 for all $D \in S^4C$.

For a divisor $D \in S^4C$ we construct the commutative diagram (2.4):

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & \mathbb{P}H^1(C,D) \cong \mathbb{P}^4 \arrow[d, "\text{Ver}"] \\ \mathbb{P}^{15} \cong \mathbb{P}\text{Ext}^1(K-D,D) \arrow[r, "\delta_D"] & \mathbb{P}\text{Sym}^2 H^1(C,D) \cong \mathbb{P}^{14}. \end{tikzcd} \tag{5.4}$$

Now $\Omega_D^0 \cong \mathbb{P}^k$ where $k = \dim(\ker(\delta_D))) - 1$. Here we introduce some notation:

$$\Sigma_k = \{D \in S^4C \mid \dim(\ker(\delta_D)) \geq k+1\}. \tag{5.5}$$

We therefore have a filtration $S^4C = \Sigma_0 \supset \Sigma_1 \cdots \supset \Sigma_n = \emptyset$ for some n. We now give a determinantal description of these Σ_k. Let π_1, π_2 be the projections from $C \times S^4C$ to the first and second factors respectively; and Δ is the universal divisor on $C \times S^4C$. We have that $\mathbb{P}\text{Ext}^1(K-D,D) \cong H^1(C, 2D-K)$. Consider the map:

$$\delta : R^1(\pi_2)_*\mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1} \longrightarrow \text{Sym}^2 R^1(\pi_2)_*\mathcal{O}(\Delta), \tag{5.6}$$

where δ is defined fibrewise by setting $\delta|_D = \delta_D : H^1(C, 2D-K) \to \text{Sym}^2 H^1(C,D)$.

Following the notation of [1] let $S^4C_l(\delta)$ be the l-th determinantal variety associated to δ. Noting from the diagram above that $16 = \text{rank}(\delta_D) + \dim(\ker(\delta_D))$ we have:

$$\begin{aligned} \Sigma_k &= \{D \in S^4C \mid \dim(\ker((\delta_D)) \geq k+1\} \\ &= \{D \in S^4C \mid \text{rank}(\delta_D) \leq 15-k\} \\ &= S^4C_{15-k}(\delta). \end{aligned}$$

By considering the commutative diagram (5.4) we can give a geometric description of $\mathrm{rank}(\delta_D)$. We know that $\mathrm{rank}(\delta_D) = \mathrm{rank}(\delta_D^\star)$, where $\delta_D^\star : \mathrm{Sym}^2 H^0(C, K-D) \to H^0(C, 2K-2D)$ is the restriction of quadrics in $\mathbb{P}H^0(C, K-D)^\star$ to the curve. Noting that $\dim(\mathrm{Sym}^2 H^0(C, K-D) = 15$, we have $\mathrm{rank}(\delta^\star) = 15 - k$ if and only if $\lambda : C \overset{|K-D|}{\longrightarrow} H^0(C, K-D)^\star$ maps C into k quadrics. Then we have:

$$\Sigma_k = S^4C_{15-k}(\delta) = \{D \in S^4C \mid \lambda(C) \text{ lies in } k \text{ independent quadrics}\} \tag{5.7}$$

Now look at the subvarieties Σ_2 and Σ_1. Our determinantal description of Σ_k tells us that:

$$\mathrm{codim}(\Sigma_k) = \mathrm{codim}(S^4C_{15-k}(\delta)) \leq (16-(15-k))(15-(15-k)) = k(k+1). \tag{5.8}$$

In particular the "expected" dimension of Σ_2 being -2 suggests that $\Sigma_2 = \emptyset$; in section 5.2 we look at this condition although we are not able to prove it. We assume from now on that $\Sigma_2 = \emptyset$. Our filtration simplifies to

$$S^4C = \Sigma_0 \supset \Sigma_1 \supset \emptyset.$$

The first step in obtaining a morphism from $\phi : S^4C \dashrightarrow \overline{\mathcal{W}^5}$ will be to blow up Σ_1. We do this by following [1] page 83. To ease the notation let

$$\begin{aligned} \mathcal{E} &:= R^1(\pi_2)_\star \mathcal{O}(2\Delta) \otimes \pi_1^\star K^{-1}, \\ \mathcal{F} &:= R^1(\pi_2)_\star \mathcal{O}(\Delta). \end{aligned} \tag{5.9}$$

We take the Grassmannian bundle $Gr(1, \mathcal{E}) \to S^4C$. Define U and Q to be the tautological bundle and quotient bundle on $Gr(1, \mathcal{E})$ respectively, then we have $0 \to U \to \pi^\star\mathcal{E} \to Q \to 0$. Now define the following map:

$$\widetilde{\delta} : U \hookrightarrow \pi^\star \mathcal{E} \overset{\pi^\star(\delta)}{\longrightarrow} \pi^\star \mathcal{F}.$$

The blow-up $\widetilde{S^4C}$ is defined to be the subvariety of $Gr(1, \mathcal{E})$ on which $\widetilde{\delta}$ vanishes.

Then

$$\begin{aligned}\widetilde{S^4C} &= \{(D,W) \mid D \in S^4C, W \text{ line in } \ker(\delta_D)\}\\ &= \{(D,E_D) \mid D \in S^4C, E_D \in \Omega^0_D\},\end{aligned}$$

where E_D denotes an extension of $\mathcal{O}(D)$. We now define a map from the blow-up to the Brill-Noether locus:

$$\widetilde{\phi} : \widetilde{S^4C} \dashrightarrow \overline{\mathcal{W}^5} \text{ where } \widetilde{\phi} : (D,E_D) \mapsto \epsilon_D(E_D), \tag{5.10}$$

where $\widetilde{\phi}$ is undefined at (D,E_D) for E_D an unstable extension.

To compute in the cohomology ring of $\widetilde{S^4C}$ we need to know the class of the determinantal subvariety Σ_1. From equation (5.8) the expected dimension of Σ_1 is 2. Under this assumption the class of Σ_1 is calculated in section 5.3.

We now identify those points of $\widetilde{S^4C}$ where $\widetilde{\phi}$ is undefined; to do this however we need the following preliminary result.

Lemma 5.1.3. *Let C be a generic curve of genus 9, and $D \in S^4C$. The curve C is birational to its image under the map $\lambda : C \xrightarrow{|K-D|} \mathbb{P}H^1(C,D) \cong \mathbb{P}^4$.*

Proof. Suppose that λ maps C $n:1$ onto a curve C_0 in $\mathbb{P}^4$ of degree d. Firstly we note that d must be at least 4; moreover if $d = 4$ then C_0 is a rational normal curve and consequently C would have a g^1_4. Hence $d \geq 5$ as a generic curve of genus 9 may not have a tetragonal pencil. The only remaining possibilities are for C_0 to be a curve of degree 6 or 12.

Suppose for a contradiction that C_0 has degree 6. Take a point p on C_0 and project from it to get the map $\pi_p : \mathbb{P}^4 \to \mathbb{P}^3$. Now consider $(\pi_p \circ \lambda) : C \overset{2:1}{\twoheadrightarrow} C_1 \subset \mathbb{P}^3$; this map is given by the linear series $|K-D-q-r|$ where $\lambda(q) = \lambda(r) = p$. Moreover, because projection from D maps both q and r to p we must have that $\mathcal{O}(D+q+r) \in W^1_6$. Then we have constructed a map $C \longrightarrow \mathbb{P}H^1(C, D+q+r)$, where $\mathcal{O}(D+q+r) \in W^1_6$. However, lemma 2.2.10 tells us that such a map is birational which contradicts the construction that $(\pi_p \circ \lambda)$ maps the curve $2:1$ onto $C_1 \subset \mathbb{P}^3$. Therefore λ must be birational. $\square$

Lemma 5.1.4. *Let C be a generic curve of genus 9, $D \in S^4C$, $E \in \Omega_D^0$, and denote the maps by $\lambda : C \xrightarrow{|K-D|} \mathbb{P}^4$ and $l : C \xrightarrow{|2K-2D|} \mathbb{P}^{15}$. Then:*

$$E \text{ is unstable } \Leftrightarrow \text{ There exists } p, q \in C \text{ such that } E \in \overline{l(p)+l(q)} \tag{5.11}$$

In general:

$$\Omega_D^0 \cap (\overline{l(p)+l(q)}) \neq \emptyset \Leftrightarrow \lambda(C) \text{ has a double point at } \lambda(p) = \lambda(q) \tag{5.12}$$

$$\Leftrightarrow \mathcal{O}(D+p+q) \in W_6^1. \tag{5.13}$$

Proof. The result of Lange-Narasimhan Lemma 2.1.2 shows that a maximal subbundle of E is of the form $K(-D-D')$ where $E \in \overline{D'}$. If we let d be the degree of D' then the degree of the maximal subbundle is $\deg(K(-D-D')) = 2g-2-4-d = 12-d$. Clearly the extension E will be unstable if and only if $d \leq 3$. Now consider the number of sections of $K(-D-D')$. The Riemann-Roch formula gives:

$$h^0(C, D+D') - h^1(C, D+D') = (4+d) - 9 + 1 = d - 4 \tag{5.14}$$

We know that $0 \to K(-D-D') \to E \to \mathcal{O}(D+D') \to 0$, by considering the long exact sequence in cohomology (and noting by Serre duality that $h^1(C, D+D') = h^0(C, K(-D-D'))$) we have the upper bound:

$$h^0(C, D+D') + h^1(C, D+D') \geq h^0(E) = 6. \tag{5.15}$$

Adding (5.14) and (5.15) we have $2h^0(C, D+D') \geq 2+d$. The condition $h^0(C, D+D') \geq [\frac{2+d}{2}]$ is obtained. In the case $d=1$ or $d=3$ we have $\mathcal{O}(D+D')$ lies in W_5^1 or W_7^2 respectively, both of which contradict the genericity of the curve since $\rho(1,5), \rho(2,7) < 0$. The only possible case is $d=2$, therefore E is unstable if and only if $E \in \overline{D'}$ where $\deg(D') = 2$. We have shown that statement (5.11) holds.

Now look at the second statement. Suppose that $\lambda(C)$ has a singularity of multiplicity 2. Note by Lemma 5.1.3 that λ is birational, so there are 2 points p, q such that $\lambda(p) = \lambda(q)$. The Veronese image of the curve will have a singularity of the same multiplicity. However, the map $l : C \to \mathbb{P}\mathrm{Ext}^1(K - D, D)$ is an embedding because the divisor $2K - 2D$ has degree 24. The commutative diagram (5.4) tells us that the singularity on $\mathrm{Ver}(\lambda(C))$ must be picked up by projecting $l(C)$ away from Ω^0_D, that is to say Ω^0_D meets $\overline{l(p) + l(q)}$. Conversely if Ω^0_D lies on a 2-secant then the resulting singularity on projection must arise from a singularity of multiplicity 2 on $\lambda(C)$. Therefore Ω^0_D meets $\overline{l(p) + l(q)}$ if and only if $\lambda(p) = \lambda(q)$, proving statement (5.12).

Finally consider (5.13). View C as embedded in canonical space $\mathbb{P}^8$, we can then use the geometric Riemann-Roch formula. Let $\mathcal{O}(D + p + q) \in W^1_6$. When adding the two points p and q to the divisor D we know that neither of them may lie in the span of D as otherwise a g^1_5 is generated which contradicts the genericity of C. However there is a relationship between the 6 points so it must be the case that the line $\overline{p + q}$ meets the span of D in a single point away from p or q. When we project from the divisor D we get the map $\lambda : C \xrightarrow{|K-D|} \mathbb{P}^4$, so we get a double point $\lambda(p) = \lambda(q)$. Working through the argument backwards shows that if $\lambda(p) = \lambda(q)$ then $\mathcal{O}(D + p + q) \in W^1_6$. $\square$

From the above lemma we conclude that $\tilde{\phi}$ is undefined at (D, E_D) if there exist p and q in C such that $\mathcal{O}(D + p + q) \in W^1_6$ and $E_D \in \overline{p + q}$. Every member of a sextic pencil $|D'| \in W^1_6$ will contain a finite number of divisors $D \in S^4C$ (generically $\binom{6}{4} = 15$), so we expect the unstable locus to be 2 dimensional.

If this subvariety of $\widetilde{S^4C}$ could be identified it would be natural to blow it up to form a morphism $\Phi : Bl(\widetilde{S^4C}) \to \overline{\mathcal{W}^5}$. If we let η_E denote the class of the fibre of Φ at a generic $E \in \overline{\mathcal{W}^5}$ and $\mathcal{O}(1)$ the hyperplane bundle on $|2\Theta|^*$ then the degree of $\overline{\mathcal{W}^5}$ is calculated in the following expression:

$$\deg(\overline{\mathcal{W}^5})\eta_E = (\Phi^*\mathcal{O}(1))^3.$$

5.2 The determinantal variety Σ_2

In equation (5.7) it was shown that:

$$\Sigma_2 = \{D \in S^4C \mid \lambda(C) \text{ lies in 2 independent quadrics}\}.$$

Therefore to show that Σ_2 is empty it is enough to prove that there is no $D \in S^4C$ such that $\lambda(C)$ lies in two independent quadrics.

Although we are not able to prove this assertion in the following section we derive some constraint conditions on the curve $\lambda(C)$ particularly in Lemma 5.2.1 and Lemma 5.2.2.

Suppose that $\lambda(C)$ lies in the intersection of quadrics Q_1 and Q_2. From now on denote $X = Q_1 \cap Q_2$. Consider the pencil of quadrics spanned by Q_1 and Q_2; all elements of this pencil contain X. Moreover, they all have rank at least 3 since $\lambda(C)$ does not lie in a hyperplane because λ is nondegenerate. Furthermore since $\lambda(C)$ is nondegenerate of degree 12 then X must be a surface.

Now consider the possibilities for X. Here we state the following result given in Cossec-Dolgachev [6] Proposition 0.3.3. Let X be a nondegenerate surface of degree 4 in $\mathbb{P}^4$. Then X is one of the following surfaces:

1. a projection of a surface of degree 4 in $\mathbb{P}^5$;
2. a cone over an elliptic quartic in a hyperplane of $\mathbb{P}^4$;
3. an anticanonical Del Pezzo surface of degree 4.

In the first case the pullback of $\lambda(C)$ to $\mathbb{P}^5$ is a nondegenerate curve of degree 12. Therefore the map from C to this curve is given by a g^5_{12}; however:

$$\rho(5, 12) = 9 - (6)(9 - 12 + 5) < 0; \tag{5.16}$$

since C is generic the first possibility cannot arise.

We now look at the second case, that X is a cone over an elliptic curve in a hyperplane of $\mathbb{P}^4$. The following lemma describes what a pencil containing X will look like.

Lemma 5.2.1. *Suppose X is contained in a pencil of quadrics, then this pencil is spanned by rank 4 quadrics. The projection of X away from the vertex is a smooth elliptic curve contained in a simple pencil of quadrics in $\mathbb{P}^3$. The projection of $\lambda(C)$ maps the curve $3:1$ onto the quartic curve.*

Proof. Let x be the vertex of X. We first show that X cannot be contained in a rank 5 quadric. Suppose for a contradiction that $X \subset Q$ for a smooth quadric Q. Consider the tangent plane to Q at x, which we denote by T_xQ. Now $T_xX \subset T_xQ$ (where T_xX is the tangent cone). Because X is a cone we have that $X \subset T_x$. Therefore we conclude that $T_xQ \cap Q \supset T_xX \cap X = X$, but $T_xQ \cap Q$ is a quadric cone in $T_xQ \cong \mathbb{P}^3$, which is a contradiction.

Note that there must exist quadrics of rank 4 in the pencil spanned by Q_1 and Q_2, otherwise the pencil would contain some quadrics of rank 2, which we may not have.

Consider a rank 4 quadric Q containing X, with vertex p. We must have that p and x coincide as otherwise $T_xX \not\subseteq T_xQ$ (see above).

We may assume Q_1 and Q_2 have rank 4. Projecting from x gives two smooth quadrics of rank 4 in $\mathbb{P}^3$ that intersect in a quartic elliptic curve, which we call C_0.

In order to show that C_0 is smooth we look at its virtual genus by viewing it as a divisor on a rank 4 quadric in $\mathbb{P}^3$ (for example the projection of Q_1). A smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; all divisors are homologous to $m_1E_0 + m_2B$, where E_0 and B are classes of lines in opposite rulings. The intersection of classes is given by $E_0.E_0 = B.B = 0$ and $E_0.B = 1$. The canonical divisor is homologous to $-2E_0 - 2B$. The virtual genus of $C_0 \sim m_1E_0 + m_2B$ is given by the following formula (see Griffiths-Harris [11] page 471)

$$\begin{aligned}\pi(C_0) &= \frac{1}{2}(C_0.C_0 + K.C_0) + 1\\ &= m_1m_2 - m_1 - m_2 + 1.\end{aligned}$$

However, $\pi(C_0)$ is at least 1, the genus of C_0 and $\deg(C_0) = 4 = m_1 + m_2$. The constraint of the above equation implies that we must have $C_0 \sim 2E_0 + 2B$ and $\pi(C_0) = 1$, the curve C_0 is smooth.

The intersection of the projected quadrics Q_1 and Q_2 is smooth of degree 4. Hence the projected quadrics intersect transversely and so span a simple pencil of quadrics containing C_0 in $\mathbb{P}^3$.

We now consider what the projection of the curve $\lambda(C)$ will look like. The curve is mapped onto C_0, consequently there is some restriction of the multiplicity of $\lambda(C)$ at the vertex depending on how many times $\lambda(C)$ is mapped onto C_0. Therefore $(\lambda(C).x) = 4r$ for $0 \leq r \leq 2$, with $\lambda(C)$ being mapped $3 - r : 1$ onto C_0. When the curve has the lowest intersection multiplicity with the vertex, projection gives a degree 8 map to $\mathbb{P}^3$. The linear series associated to this map is a g^3_8, which has Brill-Noether number $\rho(3, 8) = 9 - (4)(9 - 8 + 3) = -7$. Hence a generic curve may not intersect the vertex of the cone. Therefore projection must map the curve $3 : 1$ onto C_0. □

Now consider the third possibility, that $\lambda(C) \subset X$, an anticanonical Del Pezzo surface of degree 4. Let S be a Del Pezzo surface of degree 4, such that X is the image of the anticanonical map $p : S \xrightarrow{|-K_S|} \mathbb{P}^4$. Then S is the blowup of $\{p_1, \ldots, p_5\} \subset \mathbb{P}^2$, with exceptional divisors E_i and blowing down map $\pi : S \to \mathbb{P}^2$. A Del Pezzo surface is known as generic if none of the p_i are collinear and none are infinitely close, that is to say $p_i \in E_j$ for some $i \neq j$.

We can describe the anticanonical divisor on S:

$$-K_S = \pi^*(3H_{\mathbb{P}^2}) - \sum_{i=1}^{5} E_i, \tag{5.17}$$

where $H_{\mathbb{P}^2}$ is a hyperplane in $\mathbb{P}^2$.

Define $\widetilde{C}$ to be the proper transform of $\lambda(C)$ in S and $C_0 = \pi(\widetilde{C})$. Let d be the degree of C_0 in $\mathbb{P}^2$. We can put a lower bound on d by noting that $\lambda(C) \xrightarrow{p^{-1}} \widetilde{C} \xrightarrow{\pi} C_0$ is given by a linear series g^2_d. However, by Brill-Noether considerations a generic curve of genus 9 may only have linear series g^2_d for $d \geq 8$.

We give an expression for the virtual genus $\pi(C_0)$ of C_0, noting that $g(C_0) = g(\widetilde{C}) \geq g(\lambda(C)) = 9$

$$9 \leq \pi(C_0) = \frac{1}{2}(d-1)(d-2). \tag{5.18}$$

Let $k_i := \text{mult}_{p_i}(C_0)$. In the following Lemma a particular configuration of singularities at the p_i is taken to mean an unordered 5-tuple $(k_1, k_2, k_3, k_4, k_5)$.

Lemma 5.2.2. *Let $S \xrightarrow{\pi} \mathbb{P}^2$ be a Del Pezzo surface of degree 4 with $\{p_1, \ldots p_5\}$ the set of blow up points in $\mathbb{P}^2$. Suppose that $\lambda(C) \subset X$, the anticanonical image of S in $\mathbb{P}^4$. Then the image of $\lambda(C)$ mapped to $\mathbb{P}^2$ is constrained to have one of 7 configurations of singularities at the p_i.*

Proof. We have an expression for the proper transform $\widetilde{C}$

$$\widetilde{C} = \pi^\star(dH_{\mathbb{P}^2}) - \sum_{i=1}^{5} k_i E_i. \tag{5.19}$$

We can now give the virtual genus of $\widetilde{C}$ by using equation (5.18). Following the blowups and noting that $g(\widetilde{C}) \geq 9$, we get

$$\begin{aligned} 9 \leq \pi(\widetilde{C}) &= \pi(C_0) - \sum_{i=1}^{5} \frac{1}{2} k_i(k_i - 1) \\ &= \frac{1}{2}(d-1)(d-2) - \sum_{i=1}^{5} \frac{1}{2} k_i(k_i - 1). \end{aligned} \tag{5.20}$$

There is an upper bound on k_i because projection from p_i gives a $(d - k_i) : 1$ map from C_0 onto $\mathbb{P}^1$. For a generic curve of genus 9 the pencils have degree at least 6.

Therefore:

$$k_i \leq d - 6. \tag{5.21}$$

The degree of $\lambda(C)$ is used to find an inequality linking d and $\sum_{i=1}^{5} k_i$. We know that $\lambda(C) \subset X$ and $\deg(\lambda(C)) = 12$, so if H is a hyperplane in $\mathbb{P}^4$, then

$$\begin{aligned} 12 = (\lambda(C).H) = (\widetilde{C}. - K_S) &= (\pi^*(dH_{\mathbb{P}^2}) - \sum_{i=1}^{5} k_i E_i).(\pi^* 3H_{\mathbb{P}^2}) - \sum_{i=1}^{5} E_i) \\ &= 3d + (\sum_{i=1}^{5} k_i E_i).(\sum_{i=1}^{5} E_i). \end{aligned} \tag{5.22}$$

Note that

$$(E_i.E_j) = \begin{cases} -\quad 1 \text{ if } i = j \\ \quad\; 1 \text{ if } p_i \text{ and } p_j \text{ are infinitely close} \\ \quad\; 0 \text{ otherwise.} \end{cases}$$

Therefore equation (5.22) gives:

$$\sum_{i=1}^{5} k_i = 3d - 12 + \sum_{i<j} (k_i + k_j)(E_i.E_j) \geq 3d - 12. \tag{5.23}$$

In fact the bound $k_i \leq d - 6$ from equation (5.21) immediately gives a constraint on the Del Pezzo surfaces that $\lambda(C)$ may lie in. Suppose that at least two of the p_i (say k_4 and k_5) are infinitely close then

$$\begin{aligned} \sum_{i=1}^{5} k_i = 3d - 12 + \sum_{i<j} (k_i + k_j)(E_i.E_j) &\geq 3d - 12 + (k_4 + k_5) \\ \Rightarrow \sum_{i=1}^{3} k_i \geq 3d - 12. \end{aligned}$$

There exists a k_i which is at least the average of all the k_i; so there is a $k_i \geq d - 4$. However, from equation (5.21) we have that $k_i \leq d - 6$, a contradiction. Therefore none of the blow-up points may be infinitely close.

Hence $(E_i.E_j) = 0$ for $i \neq j$, so equation (5.23) becomes

$$\sum_{i=1}^{5} k_i = 3d - 12. \tag{5.24}$$

We now give an upper bound on d. Suppose that the Del Pezzo is generic; then the p_i are in general position and there is a conic E passing through all the p_i. This conic will be a component of C_0 if $(C_0.E) > 2d$, but $(C_0.E) \geq \sum_{i=1}^{5} k_i = 3d - 12$. Therefore, if $d \geq 13$ then E is a component of C_0. Consequently the proper transform of E will be a component of $\widetilde{C}$ which implies that $\lambda(C)$ is reducible. This is a contradiction.

If the Del Pezzo is nongeneric then three (or more) of the points p_i will be collinear. Note that we cannot have infinitely close points. If all the p_i lie on a line l then C_0 will be reducible if

$$d < (C_0.l) = \sum_{i=1}^{5} k_i = 3d - 12 \Leftrightarrow d > 6.$$

However, we already know that $d \geq 8$. Suppose now that l meets p_i $(1 \leq i \leq 4)$ and assume the most favourable condition that $k_5 = d - 6$. Then $\sum_{i=1}^{4} k_i = 2d - 6$ and C_0 is reducible if

$$d < (C_0.l) = \sum_{i=1}^{4} k_i = 2d - 6 \Leftrightarrow d < 6.$$

As before this condition is always met. If we now assume that l_1 meets p_i $(1 \leq i \leq 3)$ and l_2 meets p_4 and p_5. Then C_0 irreducible implies that

$$\begin{aligned} d \geq (C_0.l_1) = k_1 + k_2 + k_3 &\text{ and } d \geq (C_0.l_2) = k_4 + k_5 \\ \Leftrightarrow 2d &\geq \sum_{i=1}^{5} k_i = 3d - 12 \\ \Leftrightarrow d &\leq 12. \end{aligned}$$

Therefore, if $d \geq 13$ then C_0 is reducible and consequently $\lambda(C)$ is reducible. Hence $8 \leq d \leq 12$.

We now look at the values of d case by case.

Suppose $d = 8$, then (5.24) tells us that $\sum_{i=1}^{5} k_i = 3d - 12 = 12$, however the upper bound (5.21) gives $k_i \leq 2$, which is a contradiction.

When $d = 9$ we have $\sum_{i=1}^{5} k_i = 3d - 12 = 15$. From the bound (5.21) that $k_i \leq 3$ we must have $k_i = 3$. Expressing the k_i as an unordered 5-tuple we have $\underline{k_1} = (3,3,3,3,3)$. The virtual genus of $\widetilde{C}$ may be derived from equation (5.20) as

$$\pi(\widetilde{C}) = \frac{1}{2}(9-1)(9-2) - 5.\frac{1}{2}3.(3-1) = 13. \tag{5.25}$$

Consider $d = 10$. In this case $\sum_{i=1}^{5} k_i = 18$, noting that $k_i \leq 4$ we have either

$$\underline{k_2} = (4,4,4,4,2)$$
$$\underline{k_3} = (4,4,4,3,3).$$

We may again deduce the virtual genus of $\widetilde{C}$

$$\pi(\widetilde{C}) = \begin{cases} \frac{1}{2}(10-1)(10-2) - 4.\frac{1}{2}4(4-1) - \frac{1}{2}2.(2-1) & = 11 \text{ for } \underline{k_2} \\ \frac{1}{2}(10-1)(10-2) - 3.\frac{1}{2}4(4-1) - 2.\frac{1}{2}3.(3-1) & = 12 \text{ for } \underline{k_3} \end{cases} \tag{5.26}$$

For the case $d = 11$ then $\sum_{i=1}^{5} k_i = 21$ and $k_i \leq 5$. An additional restriction on the possible values of k_i is the virtual genus formula (5.20), since the genus of the curve sets a lower bound on $\pi(C_0)$. We then have

$$\sum_{i=1}^{4} \frac{1}{2}k_i(k_i - 1) \leq \frac{1}{2}(d-1)(d-2) - 9. \tag{5.27}$$

In this case $\sum_{i=1}^{5} \frac{1}{2}k_i(k_i - 1) \leq 36$. We therefore have the possibilities

$$\underline{k_4} = (5,5,5,3,3)$$
$$\underline{k_5} = (5,5,4,4,3)$$
$$\underline{k_6} = (5,4,4,4,4).$$

The different cases give the following

$$\pi(\widetilde{C}) = \begin{cases} 9 & \text{for } \underline{k_4} \\ 10 & \text{for } \underline{k_5} \\ 11 & \text{for } \underline{k_6}. \end{cases} \tag{5.28}$$

Finally if $d = 12$ then $k_i \leq 6$, however we are severely restricted by genus considerations and (5.27) gives only one possibility, $\underline{k_7} = (5, 5, 5, 5, 4)$. The virtual genus of $\widetilde{C}$ in this case is 9. This concludes our proof. □

Corollary 5.2.3. *Let S, X and C satisfy the same properties as Lemma 5.2.2. If S is nongeneric then the image of $\lambda(C)$ mapped to $\mathbb{P}^2$ is further constrained to have one of* 5 *configurations of singularities at the p_i.*

Proof. If S is nongeneric then at least three of the points p_i are collinear (recall that we cannot have points being infinitely close). Consider the $\underline{k_i}$ given above, if a line passing through three (or more) of the k_i has intersection with C_0 greater than d then $\lambda(C)$ would be reducible (see proof of Lemma 5.2.2), disallowing the configuration $\underline{k_i}$. We assume that the collinear points have the lowest k_i (and there are only three of them), in the above this equates to taking k_3, k_4 and k_5 collinear. Then configurations $\underline{k_6}$ and $\underline{k_7}$ will not be allowable. In particular, if S is nongeneric then C_0 cannot have degree 12. □

5.3 The class of Σ_1

In this section we assume that $\dim(\Sigma_1) = 2$, which is the "expected" dimension calculated by our determinantal description. The proof proceeds by using the Porteous formula to give the class of S_1 in terms of the Chern classes of $R^1(\pi_2)_*\mathcal{O}(2\Delta)\otimes\pi_1^*K^{-1}$ and $\mathrm{Sym}^2 R^1(\pi_2)_*\mathcal{O}(\Delta)$. These classes are given by a Grothendieck-Riemann-Roch calculation. However, in this process we need to determine the Chern classes of $\mathrm{Sym}^2 R^1(\pi_2)_*\mathcal{O}(\Delta)$ from $R^1(\pi_2)_*\mathcal{O}(\Delta)$. A general result that relates the Chern classes of bundles and their symmetric products is used (see Lemma 5.3.2), the proof of which is given at the end of the section.

We introduce some homology classes on S^4C. Let x be the class of the divisor $q + S^3C \subset S^4C$ and θ the pull-back of the class of the theta divisor in $H^2(J^8_C, \mathbb{Z})$.

Proposition 5.3.1. *Assuming that $dim(S_1) = 2$ then the class of Σ_1 is:*

$$22x^2 - 13x\theta + \frac{5}{2}\theta^2.$$

Proof. In our determinantal description of Σ_k we saw that Σ_1 is the 14-th determinantal variety associated to δ. To calculate the class of Σ_1 we use the Porteous formula [1] page 86, which is stated here:

Let $\phi : \mathcal{E} \to \mathcal{F}$ be a bundle map where $\mathcal{E}$ and $\mathcal{F}$ are bundles over X of ranks n and m. Then the k-th determinantal variety $X_k(\phi)$ has class:

$$\Delta_{m-k,n-k}(c_t(\mathcal{F} - \mathcal{E})) = det \begin{pmatrix} c_{m-k} & \cdots & c_{m+n-2k-1} \\ \vdots & & \vdots \\ c_{m-n+1} & \cdots & c_{m-k} \end{pmatrix}$$

where $c_i := c_i(\mathcal{F} - \mathcal{E})$

From (5.9) recall our definitions of $\mathcal{E}$ and $\mathcal{F}$:

$$\mathcal{E} = R^1(\pi_2)_\star \mathcal{O}(2\Delta) \otimes \pi_1^\star K^{-1}$$
$$\mathcal{F} = R^1(\pi_2)_\star \mathcal{O}(\Delta).$$

If we let $c_i = c_i(\mathrm{Sym}^2\mathcal{F} - \mathcal{E})$ then the class of S_1 is:

$$\Delta_{1,2}(c_t(\mathrm{Sym}^2\mathcal{F} - \mathcal{E})) = \det \begin{pmatrix} c_1 & c_2 \\ c_0 & c_1 \end{pmatrix} = c_1^2 - c_2. \tag{5.29}$$

Moreover:

$$\begin{aligned} c_1(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) &= c_1(S^2\mathcal{F}) - c_1(\mathcal{E}) \\ c_2(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) &= c_2(\mathrm{Sym}^2\mathcal{F}) - c_1(\mathrm{Sym}^2\mathcal{F})c_1(\mathcal{E}) + c_1(\mathcal{E})^2 - c_2(\mathcal{E}). \end{aligned} \tag{5.30}$$

Consequently we need to calculate the first and second Chern classes of $\mathrm{Sym}^2\mathcal{F}$ and $\mathcal{E}$.

To find $c_1(\mathrm{Sym}^2\mathcal{F})$ and $c_2(\mathrm{Sym}^2\mathcal{F})$ we calculate $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ first. We do this by using the Grothendieck-Riemann-Roch formula which states that if $\pi : X \to Y$ is a proper morphism of varieties and $\mathcal{G}$ a coherent sheaf on X, then:

$$\mathrm{ch}(\pi_! \mathcal{F}).\mathrm{td}(Y) = \pi_*(\mathrm{ch}(\mathcal{G}).\mathrm{td}(X)). \tag{5.31}$$

For a discussion of the Grothendieck-Riemann-Roch formula see [1] chapter 8. In our case we have that $X = C \times S^dC$, $Y = S^dC$, $\mathcal{G}$ a vector bundle and $\pi_1, \pi_2 = \pi$ projections to the first and second factors respectively. Suppressing the pull-backs of π_1 and π_2, the right hand side of (5.31) becomes:

$$(\pi_2)_*(\mathrm{td}(C).\mathrm{td}(S^dC).\mathrm{ch}(\mathcal{G})) = \mathrm{td}(S^dC)(\pi_2)_*(\mathrm{td}(C).\mathrm{ch}(\mathcal{G}).$$

The $\mathrm{td}(S^dC)$ factors cancel out in (5.31) and defining η to be the pull-back of a the class of a point of C to $C \times S^dC$ we obtain:

$$\mathrm{ch}((\pi_2)_! \mathcal{G}) = (\pi_2)_*((1 + (1-g)\eta).\mathrm{ch}(\mathcal{G})). \tag{5.32}$$

Note that $\mathrm{td}(C) = 1 + (1-g)\eta$.

In order to calculate $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ we set $\mathcal{G} = \mathcal{O}(\Delta)$. On page 338 of [1] it is shown that:

$$\mathrm{class}(\Delta) = \delta = d\eta + \gamma + x, \tag{5.33}$$

$$\mathrm{ch}(\mathcal{O}(\Delta)) = e^\delta = e^x + d\eta e^x - \eta\theta e^x + \gamma e^x, \tag{5.34}$$

where $\gamma \in H^1(C, \mathbb{Z}) \otimes H^1(S^dC, \mathbb{Z})$. Combining (5.32) and (5.34) we have:

$$\begin{aligned}
\mathrm{ch}((\pi_2)_! \mathcal{O}(\Delta)) &= (\pi_2)_*(1 + (1-g)\eta.(1 + d\eta - \eta\theta + \gamma)e^x) \\
&= (\pi_2)_*(1 + d\eta - \eta\theta + \gamma + (1-g)\eta)e^x \\
&= ((d-g+1) - \theta)e^x \\
&= (-4-\theta)e^x.
\end{aligned} \tag{5.35}$$

We have now determined $(\pi_2)_! \mathcal{O}(\Delta) = (\pi_2)_* \mathcal{O}(\Delta) - R^1(\pi_2)_* \mathcal{O}(\Delta)$. To find the Chern character of $\mathcal{F}$ we need to calculate $(\pi_2)_* \mathcal{O}(\Delta)$.

The fibre of $(\pi_2)_* \mathcal{O}(\Delta)$ at $D \in S^4C$ is isomorphic to $H^0(C, D)$, a one dimensional space since C has no tetragonal pencils. Consequently the direct image is a line bundle, moreover by the definition of the universal divisor Δ we know that

$$\mathcal{O}(\Delta)|_{\{p\} \times S^4C} \cong \mathcal{O}.$$

This implies that $(\pi_2)_* \mathcal{O}(\Delta) \cong \mathcal{O}$. Therefore $\mathrm{ch}((\pi_2)_* \mathcal{O}(\Delta)) = 1$; substituting this value into our expression for $(\pi_2)_! \mathcal{O}(\Delta)$ gives:

$$\begin{aligned} \mathrm{ch}(\mathcal{F}) &= \mathrm{ch}(R^1(\pi_2)_* \mathcal{O}(\Delta)) \\ &= \mathrm{ch}((\pi_2)_* \mathcal{O}(\Delta)) - \mathrm{ch}((\pi_2)_! \mathcal{O}(\Delta)) \\ &= 1 + (4 + \theta) e^x. \end{aligned} \tag{5.36}$$

We may now calculate the Chern classes from the Chern character by using Newton's formula (see Fulton [8] page 56):

$$p_n - c_1 p_{n-1} + c_2 p_{n-2} - \cdots + (-1)^{n-1} c_{n-1} p_1 + (-1)^n n c_n = 0, \tag{5.37}$$

where the p_i are defined for a bundle W by:

$$\mathrm{ch}(W) = \mathrm{rank}(W) + \sum_{n=0}^{\infty} \frac{p_n}{n!}. \tag{5.38}$$

To obtain $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ we need p_1 and p_2; from (5.36) we have:

$$\mathrm{ch}(\mathcal{F}) = 5 + (4x + \theta) + \frac{1}{2!}(4x^2 + 2x\theta) + \ldots$$

Thus:

$$\begin{aligned} c_1(\mathcal{F}) &= p_1 = 4x + \theta \\ c_2(\mathcal{F}) &= \frac{c_1(\mathcal{F})^2 - p_2}{2} = \frac{(4x+\theta)^2 - (4x^2 + 2x\theta)}{2} \\ &= 6x^2 + 3x\theta + \frac{1}{2}\theta^2. \end{aligned}$$

It is now a question of calculating $c_1(\mathrm{Sym}^2\mathcal{F})$ and $c_2(\mathrm{Sym}^2\mathcal{F})$. From Lemma (5.3.2) we have that:

$$\begin{aligned} c_1(\mathrm{Sym}^2\mathcal{F}) &= 6c_1(\mathcal{F}) \\ c_2(\mathrm{Sym}^2\mathcal{F}) &= 14c_1(\mathcal{F})^2 + 7c_2(\mathcal{F}). \end{aligned} \tag{5.39}$$

The Chern classes we need are:

$$c_1(\mathrm{Sym}^2\mathcal{F}) = 24x + 6\theta \tag{5.40}$$

$$\begin{aligned} c_2(\mathrm{Sym}^2\mathcal{F}) &= 14(4x+\theta)^2 + 7(6x^2 + 3x\theta + \frac{1}{2}\theta^2) \\ &= 266x^2 + 133x\theta + \frac{35}{2}\theta^2. \end{aligned} \tag{5.41}$$

Now we are required to find the first and second Chern classes of $\mathcal{E} = R^1(\pi_2)_\star\mathcal{O}(2\Delta)\otimes K^{-1}$. Again we use Grothendieck-Riemann-Roch; by substituting $\mathcal{G} = \mathcal{O}(2\Delta)\otimes \pi_1^\star K^{-1}$ into equation (5.32) we obtain:

$$\begin{aligned} \mathrm{ch}((\pi_2)_!\mathcal{O}(2\Delta)\otimes\pi_1^\star K^{-1}) &= (\pi_2)_\star((1+(1-g)\eta).\mathrm{ch}(\mathcal{O}(2\Delta)\otimes\pi_1^\star K^{-1})) \\ &= (\pi_2)_\star((1+(1-g)\eta).\mathrm{ch}(\mathcal{O}(2\Delta)).\mathrm{ch}(\pi_1^\star K^{-1})). \end{aligned} \tag{5.42}$$

We now calculate $\mathrm{ch}(\mathcal{O}(2\Delta))$ and $\mathrm{ch}(\pi_1^\star K^{-1})$.

We start by looking at $\pi_1^\star K^{-1}$; $c_1(K^{-1}) = 2-2g$ so $c_1(\pi_1^\star K^{-1}) = \eta(2-2g)$. We know that $\pi_1^\star K^{-1}$ is a line bundle which tells us that:

$$\mathrm{ch}(\pi_1^\star K^{-1}) = e^{c_1(\pi_1^\star K^{-1})} = e^{\eta(2-2g)}.$$

However $\eta^2 = 0$, so we simplify to:

$$\mathrm{ch}(\pi_1^\star K^{-1}) = 1 + \eta(2-2g). \tag{5.43}$$

Consider the Chern character of $\mathcal{O}(2\Theta)$. By (5.34) $\mathrm{ch}(\mathcal{O}(2\Delta)) = e^{2\delta}$ where $\delta = d\eta + \gamma + x$ by (5.33). We follow the calculation in [1] pages 338-339. Expanding e^δ

and using the relations:

$$\eta^2 = \eta\gamma = \gamma^3 = 0 \tag{5.44}$$

we obtain:

$$\begin{aligned} e^{2\delta} &= e^{2(d\eta+\gamma+x)} \\ &= \sum 2^k \frac{x^k}{k!} + \sum 2^k k \frac{d\eta x^{k-1}}{k!} + \sum 2^k \binom{k}{2} \frac{\gamma^2 x^{k-2}}{k!} + \sum 2^k k \frac{\gamma x^{k-1}}{k!}. \end{aligned}$$

Using the fact that $\gamma^2 = -2\eta\theta$ gives:

$$\begin{aligned} e^{2\delta} &= e^{2x} + 2d\eta e^{2x} - 4\eta\theta e^{2x} + 2\gamma e^{2x} \\ &= (1 + 2d\eta - 4\eta\theta + 2\gamma)e^{2x}. \end{aligned} \tag{5.45}$$

It remains now to substitute our values for $\mathrm{ch}(\pi_1^* K^{-1})$ (from (5.43)) and $\mathrm{ch}(\mathcal{O}(2\Delta))$ (from (5.45)) into (5.42):

$$\begin{aligned} \mathrm{ch}((\pi_2)_! \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}) &= (\pi_2)_*((1 + (1-g)\eta).(1 + 2d\eta - 4\eta\theta + 2\gamma).(1 + (2-2g)\eta)e^{2x}) \\ &= (\pi_2)_*((1 + (3-3g)\eta).((1 + 2d\eta - 4\eta\theta + 2\gamma))e^{2x} \\ &= (\pi_2)_*(1 + 2d\eta - 4\eta\theta + 2\gamma + 3(1-g)\eta)e^{2x} \\ &= (2d - 4\theta + 3(1-g))e^{2x} \\ &= (-16 - 4\theta)e^{2x}. \end{aligned} \tag{5.46}$$

By the definition of $\pi_!$ we have:

$$\mathrm{ch}((\pi_2)_! \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}) = \mathrm{ch}((\pi_2)_* \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}) - \mathrm{ch}(R^1(\pi_2)_* \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}).$$

However, the fibre of $(\pi_2)_* \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}$ at $D \in S^4 C$ is $H^0(C, 2D - K) = 0$, because $2D - K$ has negative degree. Therefore the higher direct image vanishes and we have:

$$\begin{aligned} \mathrm{ch}(R^1(\pi_2)_* \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}) &= -\mathrm{ch}((\pi_2)_! \mathcal{O}(2\Delta) \otimes \pi_1^* K^{-1}) \\ &= (16 + 4\theta)e^{2x}. \end{aligned} \tag{5.47}$$

To use Newton's formula (5.37)we rewrite the Chern character in the following way:

$$\mathrm{ch}(R^1(\pi_2)_*\mathcal{O}(2\Delta)\otimes\pi_1^*K^{-1}) = 16 + (32x+4\theta) + \frac{1}{2}(64x^2+16x\theta) + \ldots$$

We now give the required Chern classes:

$$c_1(\mathcal{E}) = p_1 = 32x + 4\theta \tag{5.48}$$

$$c_2(\mathcal{E}) = \frac{c_1(E)^2 - p_2}{2} = \frac{(32x+4\theta)^2 - (64x^2+16x\theta)}{2}$$
$$= 480x^2 + 120x\theta + 8\theta^2. \tag{5.49}$$

Having obtained the first and second Chern classes of $\mathcal{E}$ and $\mathrm{Sym}^2\mathcal{F}$ the class of Σ_1 can now be calculated. From equations (5.30), (5.40) and (5.48) we have:

$$c_1(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) = c_1(\mathrm{Sym}^2\mathcal{F}) - c_1(\mathcal{E}) = (24x+6\theta) - (32x+4\theta) = -8x+2\theta. \tag{5.50}$$

Furthermore (5.30), (5.41) and (5.49) give:

$$\begin{aligned} c_2(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) =& c_2(\mathrm{Sym}^2) - c_1(\mathrm{Sym}^2\mathcal{F})c_1(\mathcal{E}) + c_1(\mathcal{E})^2 - c_2(\mathcal{E}) \\ =& 266x^2 + 133x\theta + \frac{35}{2}\theta^2 - (24x+6\theta)(32x+4\theta) \\ &+ (32x+4\theta)^2 - 480x^2 - 120x\theta - 8\theta^2 \\ =& 42x^2 - 19x\theta + \frac{3}{2}\theta^2. \end{aligned}$$

The final answer is given by:

$$\begin{aligned} \mathrm{class}(\Sigma_1) &= c_1^2(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) - c_2(\mathrm{Sym}^2\mathcal{F} - \mathcal{E}) \\ &= (2\theta - 8x)^2 - (42x^2 - 19x\theta + \frac{3}{2}\theta^2) \\ &= 64x^2 - 32x\theta + 4\theta^2 - (42x^2 - 19x\theta + \frac{3}{2}\theta^2) \\ &= 22x^2 - 13x\theta + \frac{5}{2}\theta^2. \end{aligned} \tag{5.51}$$

□

Lemma 5.3.2. *If F is a vector bundle of rank r then:*

$$c_1(Sym^2F) = (r+1)c_1(F)$$
$$c_2(Sym^2F) = \frac{(r-1)(r+2)}{2}c_1(F)^2 + (r+2)c_2(F)$$

Proof. Suppose that F has Chern roots $\{\alpha_i\}_{1\le i\le r}$, then the Chern roots of Sym^2F are $\{\alpha_i+\alpha_j\}_{1\le i,j\le r}$. Therefore the Chern polynomial of Sym^2F is:

$$\prod_{i\le j}(1+(\alpha_i+\alpha_j)t) = \prod_{i=1}^{r}(1+2\alpha_i t).\prod_{i<j}(1+(\alpha_i+\alpha_j)t)$$

This tells us that:

$$c_1(\mathrm{Sym}^2F) = \sum_{i+1}^{r}2\alpha_i + \sum_{i<j}(\alpha_i+\alpha_j) = 2c_1(F)+(r-1)c_1(F) = (r+1)c_1(F)$$

and

$$c_2(\mathrm{Sym}^2F) = \sum_{i<j}4\alpha_i\alpha_j + \sum_{1\le i\le r, j<k}2\alpha_i(\alpha_j+\alpha_k) + \sum_{i<j,k<l}(\alpha_i+\alpha_j)(\alpha_k+\alpha_l). \quad (5.52)$$

To count the $\alpha_i\alpha_j$ for $i<j$ we use the symmetry of the expression and look at $\alpha_1\alpha_2$. In the first sum $\alpha_1\alpha_2$ clearly occurs 4 times, in the second sum it occurs in the form $2\alpha_1(\alpha_2+\alpha_i)$ or $2\alpha_2(\alpha_1+\alpha_i)$ - a total of $4(r-1)$ times and in the third sum we can have $(\alpha_1+\alpha_i)(\alpha_2+\alpha_j)$ where $(\alpha_1+\alpha_2)(\alpha_1+\alpha_2)$ cannot happen - $(r-1)^2-1$ overall. This gives a grand total of: $4+4(r-1)+(r-1)^2-1 = r^2+2r = r(r+2)$.

We now count the α_i^2, to do this we look at the number of times α_1^2 occurs. There are no occurences in the first sum, in the second we may have summands of the form $2\alpha_1(\alpha_1+\alpha_i)$ - which gives $2(r-1)$. Finally in the third sum we need to look at summands $(\alpha_1+\alpha_i)(\alpha_1+\alpha_j)$, since $2\le i,j\le r-1$ this amounts to choosing 2 from $r-1$ possibilities - $\binom{r-1}{2}$. We have a total of:

$$2(r-1)+\frac{(r-1)(r-2)}{2} = \frac{(r-1)(r+2)}{2}.$$

Using this information about the summands we have:

$$c_2(\mathrm{Sym}^2F) = \frac{(r-1)(r+2)}{2}\sum_{i=1}^{r}\alpha_i^2 + r(r+2)\sum_{i<j}\alpha_i\alpha_j. \quad (5.53)$$

We would like to write $c_2(\mathrm{Sym}^2 F)$ in terms of Chern classes of F. Noting that:

$$c_1(F)^2 = (\sum_{i+1}^{r} \alpha_i)^2 = \sum_{i+1}^{r} \alpha_i^2 + \sum_{i<j} 2\alpha_i\alpha_j,$$

and substituting this into (5.53) gives:

$$\begin{aligned} c_2(\mathrm{Sym}^2 F) &= \frac{(r-1)(r+2)}{2}(c_1(F)^2 - 2\sum_{i<j}\alpha_i\alpha_j) + r(r+2)\sum_{i<j}\alpha_i\alpha_j \\ &= \frac{(r-1)(r+2)}{2}c_1(F)^2 + (r+2)\sum_{i<j}\alpha_i\alpha_j \\ &= \frac{(r-1)(r+2)}{2}c_1(F)^2 + (r+2)c_2(F). \end{aligned}$$

$\square$

Bibliography

[1] E.Arbarello, M.Cornalba, P.A.Griffiths, J.Harris,*Geometry of Algebraic Curves*, Springer 1985;

[2] A.Beauville,*Fibres de rang 2 sur les courbes, fibres determinant et fonctions theta*, Bull. Soc. Math. France 116 (1988) 431-448;

[3] A.Beauville, M.S.Narasimhan, S.Ramanan,*Spectral Curves and the generalised theta divisor*, J. reine argew. Math. 398 (1989) 169-179;

[4] A.Bertram, B.Feinberg,*On stable rank two vector bundles with canonical determinant and many sections*, to appear in Proceedings of the Europroj Annual Conferences, Catania/Barcelona 1993/4;

[5] D.C.Butler,*Normal Generation of Vector Bundles over a curve*,J.Diff.Geom. 39 (1994) 1-34;

[6] F.R.Cossec, I.V.Dolgachev,*Enriques Surfaces I*, Birkhauser 1989;

[7] J.M.Drezet, M.S.Narasimhan,*Groupe de Picard des varietes de modules de fibres semi-stable sur les courbes algebrique*,Invent. Math. 97 (1989) 53-94;

[8] W.Fulton,*Intersection Theory*, Springer 1984;

[9] B.van Geemen, E.Izadi,*The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian*,Duke Algebraic Geometry preprint (alg-geom/9805037);

[10] M.Green, R. Lazarfeld,*On the projective normality of complete linear series on an algebraic curve*,Invent. Math. 83 (1985) 73-90;

[11] P.A.Griffiths, J.Harris,*Principles of Algebraic Geometry*,Wiley 1978;

[12] R.Hartshorne,*Algebraic Geometry*, Springer 1977;

[13] H.Lange, M.S.Narasimhan,*Maximal subbundles of rank two vector bundles on curves*,Math. Ann 266 (1983) 55-72;

[14] Y.Laszlo,*Un theoreme de Riemann pour les diviseurs theta sur les espaces de modules de fibres stables*,Duke Math.J. 64 (1991) 333-347;

[15] S.Mukai,*Curves, K3 surfaces and Fano 3-folds of genus* ≤ 10, in Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Kinokuniya, Tokyo (1987) 357-377;

[16] S.Mukai,*Curves and Symmetric spaces*, Proc. Japan. Acad. 68 (1992) 7-10;

[17] S.Mukai,*Curves and Grassmannians*, in Algebraic Geometry and Related Topics, eds. J-H.Yang, Y.Namikawa, K.Ueno, International Press, Boston (1993) 19-40;

[18] S.Mukai, *Vector bundles and the Brill-Noether theory*, Complex Algebraic Geometry 28 (1995) 145-158

[19] S.Mukai,*Non-Abelian Brill-Noether theory and Fano 3-folds*, preprint;

[20] M.S.Narasimhan, S.Ramanan,*Moduli of vector bundles over a compact Riemann Surface*, Ann. Math 89 (1969) 19-51;

[21] M.S.Narasimhan, C.S.Seshadri,*Stable and unitary vector bundles on a compact Riemann surface*, Ann. Math 82 (1965) 540-567;

[22] P.E.Newstead,*Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research, Bombay 1978;

[23] W.M.Oxbury, C.Pauly, E.Previato, *Subvarieties of* $\mathcal{SU}_C(2)$ *and* 2θ*-divisors in the Jacobian*, Trans. American Math. Soc. 350 (1998) 3587-3614;